

Weakly distance-regular digraphs of valency three, II

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Abstract

In this paper, we classify commutative weakly distance-regular digraphs of valency 3 with girth more than 2 and one type of arcs. As a result, commutative weakly distance-regular digraphs of valency 3 are completely determined.

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1 Introduction

Throughout this paper Γ always denotes a finite simple digraph. We write $V\Gamma$ and $A\Gamma$ for the vertex set and arc set of Γ , respectively. A *path* of length r from x to y is a sequence of vertices $(x = w_0, w_1, \dots, w_r = y)$ such that $(w_{t-1}, w_t) \in A\Gamma$ for $t = 1, 2, \dots, r$. A digraph is said to be *strongly connected* if, for any two vertices x and y , there is a path from x to y . The length of a shortest path from x to y is called the *distance* from x to y in Γ , denoted by $\partial(x, y)$. Let $\tilde{\partial}(x, y) = (\partial(x, y), \partial(y, x))$ and $\tilde{\partial}(\Gamma) = \{\tilde{\partial}(x, y) \mid x, y \in V\Gamma\}$. We call $\tilde{\partial}(x, y)$ the *two way distance* from x to y in Γ . An arc (u, v) of Γ is of *type* $(1, r)$ if $\partial(v, u) = r$. A path $(w_0, w_1, \dots, w_{r-1})$ is said to be a *circuit* of length r if $\partial(w_{r-1}, w_0) = 1$. A circuit of minimal length is said to be a *minimal circuit*. The *girth* of Γ is the length of a minimal circuit.

A strongly connected digraph Γ is said to be *weakly distance-regular* if, for all $\tilde{h}, \tilde{i}, \tilde{j} \in \tilde{\partial}(\Gamma)$ and $\tilde{\partial}(x, y) = \tilde{h}$,

$$p_{i,j}^{\tilde{h}}(x, y) := |\{z \in V\Gamma \mid \tilde{\partial}(x, z) = \tilde{i} \text{ and } \tilde{\partial}(z, y) = \tilde{j}\}|$$

depends only on $\tilde{h}, \tilde{i}, \tilde{j}$. The integers $p_{i,j}^{\tilde{h}}$ are called the *intersection numbers*. We say that Γ is *commutative* if $p_{i,j}^{\tilde{h}} = p_{j,i}^{\tilde{h}}$ for all $\tilde{i}, \tilde{j}, \tilde{h} \in \tilde{\partial}(\Gamma)$. A weakly distance-regular digraph is *thin* (resp. *quasi-thin*) if the maximum value of its intersection numbers is 1 (resp. 2).

Some special families of weakly distance-regular digraphs were classified. See [8, 7] for valency 2, [7] for thin case and [11] for quasi-thin case under the assumption

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of the commutativity. Also in [7], Suzuki proposed the project to classify weakly distance-regular digraphs of valency 3. In [9], Wang determined all such digraphs with girth 2 under the assumption of the commutativity. In [10], we classified these digraphs with girth more than 2 and two types of arcs. In this paper, we continue this project, and obtain the following result.

Theorem 1.1 *Let Γ be a commutative weakly distance-regular digraph of valency 3 and girth more than 2. If Γ has one type of arcs, then Γ is isomorphic to one of the following digraphs:*

- (i) $\text{Cay}(\mathbb{Z}_7, \{1, 2, 4\})$.
- (ii) $\text{Cay}(Q_8, \{i, j, k\})$, Q_8 is the quaternion group of order 8.
- (iii) $\text{Cay}(\mathbb{Z}_{13}, \{1, 3, 9\})$.
- (iv) The eighteenth digraph with 18 vertices in [3].
- (v) $\text{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\})$.
- (vi) $\text{Cay}(\mathbb{Z}_g \times \mathbb{Z}_3, \{(1, 0), (1, 1), (1, 2)\})$, $g \geq 3$.
- (vii) $\text{Cay}(\mathbb{Z}_n \times \mathbb{Z}_n, \{(1, 0), (0, 1), (n-1, n-1)\})$, $n \notin 3\mathbb{Z} \setminus \{3\}$.
- (viii) $\text{Cay}(\mathbb{Z}_n \times \mathbb{Z}_{3n}, \{(0, 1), (1, 1), (n-1, 3n-2)\})$, $n \geq 2$.

Routinely, all digraphs in above theorem are commutative weakly distance-regular. The digraph in (iv) is not a Cayley digraph by [4]. For the last three families of Cayley digraphs, in Table 1, we list the two way distance from the identity element to any other element of the corresponding group.

2 Basic results

In this section, we introduce terminology and a basic result which are used frequently in this paper, and show the structure of the proof of our main result.

Let Γ be a commutative weakly distance-regular digraph and $R = \{\Gamma_{\tilde{i}} \mid \tilde{i} \in \tilde{\partial}(\Gamma)\}$, where $\Gamma_{\tilde{i}} = \{(x, y) \in V\Gamma \times V\Gamma \mid \tilde{\partial}(x, y) = \tilde{i}\}$. Then $(V\Gamma, R)$ is a commutative association scheme ([2, 12]).

For two nonempty subsets E and F of R , define

$$EF := \{\Gamma_{\tilde{h}} \mid \sum_{\Gamma_{\tilde{i}} \in E} \sum_{\Gamma_{\tilde{j}} \in F} p_{\tilde{i}, \tilde{j}}^{\tilde{h}} \neq 0\},$$

and write $\Gamma_{\tilde{i}}\Gamma_{\tilde{j}}$ instead of $\{\Gamma_{\tilde{i}}\}\{\Gamma_{\tilde{j}}\}$. The size of $\Gamma_{\tilde{i}}(x) := \{y \in V\Gamma \mid \tilde{\partial}(x, y) = \tilde{i}\}$ depends only on \tilde{i} , denoted by $k_{\tilde{i}}$. For any $(a, b) \in \tilde{\partial}(\Gamma)$, we usually write $k_{a,b}$ (resp. $\Gamma_{a,b}$) instead of $k_{(a,b)}$ (resp. $\Gamma_{(a,b)}$).

Lemma 2.1 ([2, Chapter II, Proposition 2.2] and [1, Proposition 5.1]) *For each $\tilde{i} := (a, b) \in \tilde{\partial}(\Gamma)$, define $\tilde{i}^* = (b, a)$. The following hold:*

- (i) $k_{\tilde{a}} k_{\tilde{e}} = \sum_{\tilde{f} \in \tilde{\partial}(\Gamma)} p_{\tilde{a}, \tilde{e}}^{\tilde{f}} k_{\tilde{f}}$.

- (ii) $p_{\tilde{d},\tilde{e}}^{\tilde{f}} k_{\tilde{f}} = p_{\tilde{f},\tilde{e}^*}^{\tilde{d}} k_{\tilde{d}} = p_{\tilde{d}^*,\tilde{f}}^{\tilde{e}} k_{\tilde{e}}.$
- (iii) $|\Gamma_{\tilde{d}} \Gamma_{\tilde{e}}| \leq \gcd(k_{\tilde{d}}, k_{\tilde{e}}).$
- (iv) $\sum_{\tilde{e} \in \tilde{\partial}(\Gamma)} p_{\tilde{d},\tilde{e}}^{\tilde{f}} = k_{\tilde{d}}.$
- (v) $\sum_{\tilde{f} \in \tilde{\partial}(\Gamma)} p_{\tilde{d},\tilde{e}}^{\tilde{f}} p_{\tilde{g},\tilde{f}}^{\tilde{h}} = \sum_{\tilde{l} \in \tilde{\partial}(\Gamma)} p_{\tilde{g},\tilde{d}}^{\tilde{l}} p_{\tilde{l},\tilde{e}}^{\tilde{h}}.$
- (vi) $\text{lcm}(k_{\tilde{d}}, k_{\tilde{e}}) \mid p_{\tilde{d},\tilde{e}}^{\tilde{f}} k_{\tilde{f}}.$

In the remaining of this paper, we always assume that Γ is a commutative weakly distance-regular digraph of valency 3 satisfying $k_{1,g-1} = 3$, where $g \geq 3$. For $\tilde{i}, \tilde{j} \in \tilde{\partial}(\Gamma)$ and $x, y \in V\Gamma$, we set $P_{\tilde{i},\tilde{j}}(x, y) := \Gamma_{\tilde{i}}(x) \cap \Gamma_{\tilde{j}^*}(y).$

Proposition 2.2 *If $p_{(1,g-1),(1,g-1)}^{(2,g-2)} = 3$, then Γ is isomorphic to one of the digraphs in Theorem 1.1 (vi).*

Proof. For fixed vertex $x_{0,0} \in V\Gamma$, let $(x_{0,0}, x_{1,0}, \dots, x_{g-1,0})$ be a minimal circuit, where the first subscription of x are taken modulo g . Assume that $\Gamma_{1,g-1}(x_{i,0}) = \{x_{i+1,0}, x_{i+1,1}, x_{i+1,2}\}$ for $0 \leq i \leq g-1$. The fact $p_{(1,g-1),(1,g-1)}^{(2,g-2)} = 3$ implies that $P_{(1,g-1),(1,g-1)}(x_{i,0}, x_{i+2,0}) = \Gamma_{1,g-1}(x_{i,0})$ and $(x_{i-1,0}, x_{i,0}, x_{i+1,j}, x_{i+2,0}, \dots, x_{i-2,0})$ is a minimal circuit for $0 \leq j \leq 2$. By $\tilde{\partial}(x_{i-1,0}, x_{i+1,j}) = (2, g-2)$, we obtain $\partial(x_{i,j'}, x_{i+1,j}) = 1$ for $0 \leq j' \leq 2$. Then $\Gamma_{1,g-1}(x_{i,j}) = \{x_{i+1,0}, x_{i+1,1}, x_{i+1,2}\}$ and $\Gamma_{g-1,1}(x_{i,j}) = \{x_{i-1,0}, x_{i-1,1}, x_{i-1,2}\}$. Since $(x_{i,j}, x_{i+1,j'}, \dots, x_{i',j'}, x_{i'+1,j}, \dots, x_{i-1,j})$ is a minimal circuit with $i \neq i'$, one gets $x_{i,j} \neq x_{i',j'}$, which implies $|V\Gamma| = 3g$. The desired result holds. \square

Lemma 2.3 *If $p_{(1,g-1),(1,g-1)}^{(2,g-2)} = 2$, then the girth of Γ is 3.*

Proof. For fixed $x_{0,0} \in V\Gamma$, let $(x_{0,0}, x_{1,0}, \dots, x_{g-1,0})$ be a minimal circuit and $\Gamma_{1,g-1}(x_{0,0}) = \{x_{1,0}, x_{1,1}, x_{1,2}\}$. Since $p_{(1,g-1),(1,g-1)}^{(2,g-2)} = 2$, from Lemma 2.1 (ii), we have $p_{(2,g-2),(g-1,1)}^{(1,g-1)} = 2$ and $k_{2,g-2} = 3$. Without loss of generality, we may assume that $\Gamma_{2,g-2}(x_{0,0}) = \{x_{2,0}, x_{2,1}, x_{2,2}\}$ and $x_{2,i}, x_{2,i+1} \in \Gamma_{1,g-1}(x_{1,i})$, where $x_{2,3} = x_{2,0}$ and $i = 0, 1, 2$. Since $x_{1,0} \in \Gamma_{g-2,2}(x_{3,0})$, there exists a vertex $x_0 \in P_{(1,g-1),(1,g-1)}(x_{1,0}, x_{3,0}) \setminus \{x_{2,0}\}$. By $x_0 \in \Gamma_{2,g-2}(x_{0,0}) = \{x_{2,0}, x_{2,1}, x_{2,2}\}$, we obtain $x_0 = x_{2,1}$. Since $p_{(2,g-2),(g-1,1)}^{(1,g-1)} = 2$, we get $(x_{g-1,0}, x_{1,1}) \notin \Gamma_{2,g-2}$, which implies $x_{g-1,0} = x_{2,0}$. Thus, $g = 3$. \square

Let $A_{i,j}$ denote a matrix with rows and columns indexed by $V\Gamma$ such that $(A_{i,j})_{x,y} = 1$ if $\tilde{\partial}(x, y) = (i, j)$, and $(A_{i,j})_{x,y} = 0$ otherwise.

Proposition 2.4 *Suppose $p_{(1,2),(1,2)}^{(2,1)} = 2$. If $|\Gamma_{1,2} \Gamma_{2,1}| = 3$, then Γ is isomorphic to one of the digraphs in Theorem 1.1 (i) and (iii).*

Proof. We claim $p_{(1,2),(2,1)}^{(2,2)} = 0$. Suppose not. Let x', y', z' be three vertices such that $\partial(x', y') = \partial(z', y') = 1$ and $\tilde{\partial}(x', z') = (2, 2)$. By Lemma 2.1 (i) and (vi), we have $\Gamma_{1,2}^2 = \{\Gamma_{2,1}, \Gamma_{2,2}\}$, which implies that there exist two vertices $w' \in P_{(1,2),(1,2)}(z', x')$ and $w'' \in P_{(1,2),(1,2)}(x', z')$. Since $p_{(2,1),(2,1)}^{(1,2)} = p_{(1,2),(1,2)}^{(2,1)} = 2$, one gets w'' or $y' \in \Gamma_{2,1}(w')$. By $z' \in P_{(1,2),(1,2)}(w'', w')$ or $y' \in P_{(1,2),(1,2)}(z', w')$, we obtain $\Gamma_{1,2} \in \Gamma_{1,2}^2$, a contradiction. Thus, our claim is valid.

By Lemma 2.1 (i) and the claim, we have $A_{1,2}A_{2,1} = 3A_0 + p_{(1,2),(2,1)}^{(h,l)}A_{h,l} + p_{(1,2),(2,1)}^{(l,h)}A_{l,h}$ with $h < l$ and $(h, l) \in \{(1, 2), (2, 3)\}$.

Case 1. $(h, l) = (1, 2)$.

By Lemma 2.1 (i), $\Gamma_{1,2}^2 = \{\Gamma_{1,2}, \Gamma_{2,1}\}$. In view of the commutativity of Γ , we have $\Gamma_{1,2}^i = \{\Gamma_{0,0}, \Gamma_{1,2}, \Gamma_{2,1}\}$ for $i \geq 3$. Hence, $\tilde{\partial}(\Gamma) = \{(0, 0), (1, 2), (2, 1)\}$ and $|V\Gamma| = 7$. From [3], $\Gamma \simeq \text{Cay}(\mathbb{Z}_7, \{1, 2, 4\})$.

Case 2. $(h, l) = (2, 3)$.

By Lemma 2.1 (i) and (vi), we get $A_{1,2}^2 = 2A_{2,1} + p_{(1,2),(1,2)}^{(2,3)}A_{2,3}$. Since $p_{(1,2),(2,1)}^{(2,3)} \neq 0$, from Lemma 2.1 (iv), one has $p_{(1,2),(1,2)}^{(2,3)} = 1$ and $k_{2,3} = 3$. Then $p_{(1,2),(2,1)}^{(2,3)} = 1$.

Pick a path (x, y, z) such that $\tilde{\partial}(x, z) = (2, 3)$. Suppose $\Gamma_{1,2}(z) = \{w, w_1, w_2\}$. By $p_{(2,1),(2,1)}^{(1,2)} = p_{(1,2),(1,2)}^{(2,1)} = 2$, we may assume $\tilde{\partial}(y, w_1) = \tilde{\partial}(y, w_2) = (2, 1)$. Since $p_{(1,2),(2,1)}^{(2,3)}k_{2,3} = 3$, from Lemma 2.1 (ii), we get $p_{(2,3),(1,2)}^{(1,2)} = p_{(3,2),(1,2)}^{(1,2)} = 1$. Without loss of generality, we may assume $\tilde{\partial}(x, w_1) = \tilde{\partial}(w_2, x) = (3, 2)$. Since $p_{(1,2),(2,1)}^{(2,3)} = 1$, we have $w \in P_{(1,2),(2,1)}(x, z)$, which implies $\Gamma_{2,3}\Gamma_{1,2} = \{\Gamma_{2,3}, \Gamma_{3,2}, \Gamma_{1,2}\}$ and $\Gamma_{1,2}^3 = \{\Gamma_{2,3}, \Gamma_{3,2}, \Gamma_{0,0}, \Gamma_{1,2}\}$.

Since $p_{(1,2),(1,2)}^{(2,3)} = 1$, we may assume that $\Gamma_{1,2}(w_1) = \{y, y_1, y_2\}$ and $y_1 \in P_{(1,2),(1,2)}(w_1, x)$. By $p_{(3,2),(2,1)}^{(3,2)} = p_{(2,3),(1,2)}^{(2,3)} \neq 0$, $y_2 \in P_{(3,2),(2,1)}(x, w_1)$. Hence, $\Gamma_{3,2}\Gamma_{1,2} = \{\Gamma_{3,2}, \Gamma_{2,1}, \Gamma_{1,2}\}$. It follows that $\Gamma_{1,2}^4 = \{\Gamma_{2,3}, \Gamma_{3,2}, \Gamma_{1,2}, \Gamma_{2,1}\}$ and $\Gamma_{1,2}^i = \{\Gamma_{0,0}, \Gamma_{1,2}, \Gamma_{2,1}, \Gamma_{2,3}, \Gamma_{3,2}\}$ for $i \geq 5$. Then $\tilde{\partial}(\Gamma) = \{(0, 0), (1, 2), (2, 1), (2, 3), (3, 2)\}$ and $|V\Gamma| = 13$. By [3], $\Gamma \simeq \text{Cay}(\mathbb{Z}_{13}, \{1, 3, 9\})$. \square

In the following of this paper, we may assume $s = \max\{j \mid p_{(1,g-1),(1,g-1)}^{(i,j)} \neq 0\}$.

Lemma 2.5 Suppose $p_{(1,2),(1,2)}^{(2,1)} = 2$ and $|\Gamma_{1,2}\Gamma_{2,1}| = 2$. The following hold:

- (i) If $p_{(1,2),(1,2)}^{(2,s)} = 1$, then $A_{1,2}A_{2,1} = 3A_{0,0} + A_{3,3}$ and $k_{3,3} = 6$.
- (ii) If $p_{(1,2),(1,2)}^{(2,s)} = 3$, then $A_{1,2}A_{2,1} = 3A_{0,0} + 2A_{3,3}$ and $k_{3,3} = 3$.

Proof. Let $A_{1,2}A_{2,1} = 3A_{0,0} + p_{(1,2),(2,1)}^{(h,h)}A_{h,h}$ with $2 \leq h \leq 3$. By Lemma 2.1 (i) and (vi), one has $A_{1,2}^2 = 2A_{2,1} + p_{(1,2),(1,2)}^{(2,s)}A_{2,s}$ with $s > 1$. It follows from Lemma 2.1 (v) that $p_{(1,2),(1,2)}^{(2,1)}p_{(2,1),(2,1)}^{(1,2)} + p_{(1,2),(1,2)}^{(2,s)}p_{(2,1),(2,s)}^{(1,2)} = 3 + p_{(2,1),(1,2)}^{(h,h)}p_{(h,h),(1,2)}^{(1,2)}$. In view of Lemma 2.1 (i), we obtain $p_{(1,2),(2,1)}^{(h,h)}k_{h,h} = 6$ and $p_{(1,2),(1,2)}^{(2,s)}k_{2,s} = 3$, which imply $p_{(h,h),(1,2)}^{(1,2)} = p_{(2,1),(2,1)}^{(1,2)} = 2$ and $p_{(2,1),(2,s)}^{(1,2)} = 1$ from Lemma 2.1 (ii). Hence, $2p_{(2,1),(1,2)}^{(h,h)} = p_{(1,2),(1,2)}^{(2,s)} + 1$. By the commutativity of Γ , we have $(h, h) \neq (2, s)$. Thus,

$h = 3$. If $p_{(1,2),(1,2)}^{(2,s)} = 1$, then $p_{(2,1),(1,2)}^{(3,3)} = 1$; if $p_{(1,2),(1,2)}^{(2,s)} = 3$, then $p_{(2,1),(1,2)}^{(3,3)} = 2$. The desired results hold. \square

In Section 3, we prove our main result under the assumption that $p_{(1,2),(1,2)}^{(2,1)} = 2$ and $|\Gamma_{1,2}\Gamma_{2,1}| = 2$, and divide it into two cases base on Lemma 2.5.

Proposition 2.6 *Suppose $p_{(1,g-1),(1,g-1)}^{(2,g-2)} = 1$ and $\Gamma_{1,g-1} \in \Gamma_{1,g-1}^2$. Then Γ is isomorphic to the digraph in Theorem 1.1 (ii).*

Proof. By Lemma 2.1 (i), we get $A_{1,g-1}A_{g-1,1} = 3A_{0,0} + A_{1,g-1} + A_{g-1,1}$. Pick distinct vertices x, y, z, w such that $\partial(x, y) = \partial(y, z) = \partial(x, z) = \partial(y, w) = 1$ and $\tilde{\partial}(x, w) = (2, g-2)$. Observe $y \in P_{(g-1,1),(1,g-1)}(z, w)$. In view of the commutativity of Γ and $p_{(1,g-1),(1,g-1)}^{(2,g-2)} = 1$, one has $\partial(w, z) = 1$. Since $z \in P_{(1,g-1),(g-1,1)}(x, w)$ and $\partial(x, w) = 2$, we obtain $\partial(w, x) = 1$ and $g = 3$.

Let w' be the vertex such that $\Gamma_{1,2}(y) = \{z, w, w'\}$. Since $p_{(1,2),(1,2)}^{(1,2)} \neq 0$, we have $w' \in P_{(1,2),(1,2)}(y, w)$ and $z \in P_{(1,2),(1,2)}(y, w')$. By $p_{(2,1),(2,1)}^{(1,2)} = p_{(1,2),(1,2)}^{(2,1)} = 1$, one gets $(x, w') \in \Gamma_{2,2}$ and $\Gamma_{1,2}^2 = \{\Gamma_{1,2}, \Gamma_{2,1}, \Gamma_{2,2}\}$. Since $z, y \in P_{(1,2),(1,2)}(x, w')$, from Lemma 2.1 (i) and (vi), we obtain $p_{(1,2),(1,2)}^{(2,2)} = 3$ and $k_{2,2} = 1$. Lemma 2.1 (iii) implies that $\Gamma_{2,2}\Gamma_{1,2} = \{\Gamma_{2,1}\}$ and $\Gamma_{1,2}^i = \{\Gamma_{1,2}, \Gamma_{2,1}, \Gamma_{0,0}, \Gamma_{2,2}\}$ for $i \geq 3$. Then $\tilde{\partial}(\Gamma) = \{(0, 0), (1, 2), (2, 1), (2, 2)\}$ and $|\text{VT}| = 8$. By [3], the desired result follows. \square

Lemma 2.7 *Suppose $p_{(1,g-1),(1,g-1)}^{(2,g-2)} = 1$ and $\Gamma_{1,g-1} \notin \Gamma_{1,g-1}^2$. Then $k_{2,s} = 1$ or 3. Moreover, the following hold:*

- (i) *If $k_{2,s} = 3$, then $A_{1,g-1}^2 = A_{2,g-2} + 2A_{2,s}$.*
- (ii) *If $k_{2,s} = 1$, then $A_{1,g-1}^2 = A_{2,g-2} + 3A_{2,s}$ or $A_{1,g-1}^2 = A_{2,g-2} + A_{2,l} + 3A_{2,s}$ for $g-2 < l < s$.*
- (iii) *$p_{(1,g-1),(g-1,1)}^{\tilde{h}} \leq 1$ for $\tilde{h} \neq (0, 0)$.*

Proof. We claim that $k_{2,s} \neq 2$ and $k_{2,l} \neq 1$ with $g-2 < l < s$. Suppose $k_{2,s} = 2$. Let (x, y_0, z_0, w) be a path such that $\tilde{\partial}(x, z_0) = (2, g-2)$ and $\tilde{\partial}(y_0, w) = (2, s)$. By Lemma 2.1 (ii), we obtain $p_{(1,g-1),(1,g-1)}^{(2,s)} = 3$ and $p_{(g-1,1),(2,s)}^{(1,g-1)} = 2$, which imply that there exist two vertices $z_1 \in P_{(1,g-1),(1,g-1)}(y_0, w) \setminus \{z_0\}$ and $y_1 \in P_{(g-1,1),(2,s)}(z_1, w) \setminus \{y_0\}$. Hence, $\tilde{\partial}(x, z_1) = (2, s)$ and $\partial(x, y_1) = \partial(y_1, z_0) = 1$. Observe $y_0, y_1 \in P_{(1,g-1),(1,g-1)}(x, z_0)$, contrary to $p_{(1,g-1),(1,g-1)}^{(2,g-2)} = 1$. Suppose $k_{2,l} = 1$. Let (x_0, x_1, x_2, x_3) be a path such that $\tilde{\partial}(x_0, x_2) = (2, l)$ and $\tilde{\partial}(x_1, x_3) = (2, s)$. It follows from Lemma 2.1 (iii) that $|\Gamma_{2,l}\Gamma_{1,g-1}| = 1$ and $\partial(x_3, x_0) = l-1$. Hence, $s = \partial(x_3, x_1) \leq \partial(x_3, x_0) + 1 = l$, contrary to $l < s$. Thus, our claim is valid.

Observe $p_{(g-1,1),(1,g-1)}^{(0,0)} = 3$ and $|\Gamma_{g-1,1}\Gamma_{1,g-1}| \geq 2$. By Lemma 2.1 (v), we have $\sum_{\tilde{i} \in \tilde{\partial}(\Gamma)} p_{(1,g-1),(1,g-1)}^{\tilde{i}} p_{(g-1,1),\tilde{i}}^{(1,g-1)} = \sum_{\tilde{j} \in \tilde{\partial}(\Gamma)} p_{(g-1,1),(1,g-1)}^{\tilde{j}} p_{\tilde{j},(1,g-1)}^{(1,g-1)} > 3$. It follows from Lemma 2.1 (iv) that $p_{(1,g-1),(1,g-1)}^{\tilde{i}} \geq 2$ for some $\tilde{i} \in \tilde{\partial}(\Gamma)$.

Since $p_{(1,g-1),(1,g-1)}^{(2,g-2)} = 1$, from Lemma 2.1 (iii), we have $|\Gamma_{1,g-1}^2| = 2$ or 3. If $|\Gamma_{1,g-1}^2| = 2$, then $p_{(1,g-1),(1,g-1)}^{(2,s)} = 2$ or 3, which implies $k_{2,s} = 3$ or 1 from Lemma 2.1 (vi) and the claim; if $|\Gamma_{1,g-1}^2| = 3$, then $p_{(1,g-1),(1,g-1)}^{(2,l)} = 1$, $p_{(1,g-1),(1,g-1)}^{(2,s)} = 3$ and $k_{2,s} = 1$. Thus, (i) and (ii) hold.

We conclude $\sum_{\tilde{j} \in \tilde{\partial}(\Gamma)} p_{(g-1,1),(1,g-1)}^{\tilde{j}} p_{\tilde{j},(1,g-1)}^{(1,g-1)} = \sum_{\tilde{i} \in \tilde{\partial}(\Gamma)} p_{(1,g-1),(1,g-1)}^{\tilde{i}} p_{(g-1,1),\tilde{i}}^{(1,g-1)} = 5$ from Lemma 2.1 (i) and (ii). Note that $|\Gamma_{1,g-1}\Gamma_{g-1,1}| = 2$ or 3. If $\Gamma_{1,g-1}\Gamma_{g-1,1} = \{\Gamma_{0,0}, \Gamma_{\tilde{h}}\}$, then $p_{\tilde{h},(1,g-1)}^{(1,g-1)} k_{1,g-1} = p_{(g-1,1),(1,g-1)}^{\tilde{h}} k_{\tilde{h}} = 6$, which implies $p_{\tilde{h},(1,g-1)}^{(1,g-1)} = 2$ and $p_{(g-1,1),(1,g-1)}^{\tilde{h}} = 1$. If $\Gamma_{1,g-1}\Gamma_{g-1,1} = \{\Gamma_{0,0}, \Gamma_{\tilde{h}_1}, \Gamma_{\tilde{h}_2}\}$, by Lemma 2.1 (vi), then $p_{\tilde{h}_j,(1,g-1)}^{(1,g-1)} k_{1,g-1} = p_{(g-1,1),(1,g-1)}^{\tilde{h}_j} k_{\tilde{h}_j} = 3$ for $j = 1, 2$, which implies $p_{\tilde{h}_j,(1,g-1)}^{(1,g-1)} = 1$ and $p_{(g-1,1),(1,g-1)}^{\tilde{h}_j} = 1$. Thus, (iii) is valid. \square

In Section 4, we prove Theorem 1.1 under the assumption that $p_{(1,g-1),(1,g-1)}^{(2,g-2)} = 1$ and $\Gamma_{1,g-1} \notin \Gamma_{1,g-1}^2$, and divide it into two cases base on Lemma 2.7.

3 $p_{(1,2),(1,2)}^{(2,1)} = 2$ and $|\Gamma_{1,2}\Gamma_{2,1}| = 2$

In this case, based on Lemma 2.5, we prove our main result according to two separate assumptions. First, we introduce a concept of h -chain which is useful in the proof under the second assumption.

In Γ , we call that a path $L := (y_0, y_1, \dots, y_j)$ is an h -chain if $\tilde{\partial}(y_i, y_{i+2}) = (2, h)$ for $0 \leq i \leq j-2$. The h -chain L is *closed* if $y_0 = y_j$. If L is a shortest closed h -chain, we say that L is an h -line. A concept of chain was given to deal with the classification of primitive association schemes [5, 6].

For $\tilde{f} \in \tilde{\partial}(\Gamma)$, we define a relation $\tilde{\Gamma}_{\tilde{f}}$ on the set of h -chains as follows. For any two h -chains $L = (y_0, y_1, \dots, y_j)$ and $L' = (y'_0, y'_1, \dots, y'_{j'})$, $(L, L') \in \tilde{\Gamma}_{\tilde{f}}$ if and only if $j = j'$ and $\tilde{\partial}(y_i, y'_i) = \tilde{f}$ for $0 \leq i \leq j$.

Suppose that the length of any h -line is a constant m . Let L_0, L_1, \dots, L_j be h -lines. We say that (L_0, L_1, \dots, L_j) is a *chain of h -lines* if $(L_i, L_{i+1}) \in \tilde{\Gamma}_{1,g-1}$ for $0 \leq i \leq j-1$, and $(L_{i'}, L_{i'+2}) \in \tilde{\Gamma}_{2,h}$ for $0 \leq i' \leq j-2$. In particular, we say that a chain of h -lines (L_0, L_1, \dots, L_j) is an h -plane if $j = m$.

By Lemma 2.1 (i) and (vi), we have $A_{1,2}^2 = 2A_{2,1} + p_{(1,2),(1,2)}^{(2,s)} k_{2,s}$ with $s > 1$, which implies $p_{(1,2),(1,2)}^{(2,s)} = 1$ or 3.

Proposition 3.1 *If $p_{(1,2),(1,2)}^{(2,s)} = 3$, then $\Gamma \simeq \text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_6, \{(0, 1), (1, 1), (1, 4)\})$.*

Proof. Note that $2 \leq s \leq 4$. By Lemma 2.1 (i), $k_{2,s} = 1$. Pick a path (x, y, z) such that $\tilde{\partial}(x, z) = (2, s)$. It follows from Lemma 2.1 (iii) that $|\Gamma_{2,s}\Gamma_{1,2}| = 1$. Since $p_{(2,1),(2,1)}^{(1,2)} \neq 0$, there exists a vertex $w \in P_{(2,1),(2,1)}(y, z)$ such that $\partial(w, x) = s-1$. By Lemma 2.5 (ii), we have $A_{1,2}A_{2,1} = 3A_{0,0} + 2A_{3,3}$ with $k_{3,3} = 3$, which implies $\tilde{\partial}(x, w) = (3, 3)$ and $s = 4$ from $y \in P_{(1,2),(2,1)}(x, w)$. Hence, $\Gamma_{2,4}\Gamma_{1,2} = \{\Gamma_{3,3}\}$. In view of the commutativity of Γ , we get $\Gamma_{1,2}^3 = \{\Gamma_{0,0}, \Gamma_{3,3}\}$.

Let $\Gamma_{1,2}(w) = \{y, y_1, y_2\}$. Since $\tilde{\partial}(x, w) = (3, 3)$ and $p_{(2,1),(1,2)}^{(3,3)} = 2$, there exist two vertices z_1, z_2 such that $\{z_1, z_2\} = P_{(2,1),(1,2)}(x, w)$. For each $i = 1, 2$, we have $\tilde{\partial}(z_{j_i}, y_i) = (2, 1)$ for some $j_i \in \{1, 2\}$ by $p_{(2,1),(2,1)}^{(1,2)} = p_{(1,2),(1,2)}^{(2,1)} = 2$. The fact that $p_{(1,2),(1,2)}^{(1,2)} = 0$ implies $\partial(y_i, x) = 2$. Since $p_{(1,2),(2,1)}^{(3,3)} = 2$, one gets $\Gamma_{3,3}\Gamma_{1,2} = \{\Gamma_{1,2}, \Gamma_{4,2}\}$. Hence, $\Gamma_{1,2}^4 = \{\Gamma_{1,2}, \Gamma_{4,2}\}$.

Since $k_{4,2} = 1$, from Lemma 2.1 (iii), we get $\Gamma_{4,2}\Gamma_{1,2} = \{\Gamma_{2,1}\}$, which implies $\Gamma_{1,2}^5 = \{\Gamma_{2,1}, \Gamma_{2,4}\}$. Hence, $\Gamma_{1,2}^{j+3i} = \Gamma_{1,2}^j$ for $j = 2, 3, 4$ and $i \geq 0$. Then $\tilde{\partial}(\Gamma) = \{(0, 0), (1, 2), (2, 1), (2, 4), (3, 3), (4, 2)\}$ and $|V\Gamma| = 12$. The desired result follows from [3]. \square

In the following of this section, we may assume $p_{(1,2),(1,2)}^{(2,s)} = 1$, and we characterize some relations by length of an s -chain, and give a construction of the digraph Γ .

By Lemma 2.5 (i), we have $A_{1,2}A_{2,1} = 3A_{0,0} + A_{3,3}$ with $k_{3,3} = 6$. Since $A_{1,2}^2 = 2A_{2,1} + A_{2,s}$, from Lemma 2.1 (i), we get $k_{2,s} = 3$. For $i \geq 1$, we set

$$\Gamma_i := \{(x, y) \mid \text{there exists an } s\text{-chain from } x \text{ to } y \text{ of length } i\},$$

$\Gamma_i(x) := \{y \mid (x, y) \in \Gamma_i\}$ and $k_i(x) = |\Gamma_i(x)|$. Let $k_i = k_i(x)$, when $k_i(x)$ depends only on i and does not depend on the choice of x . Note that $\Gamma_1 = \Gamma_{1,2}$ and $\Gamma_2 = \Gamma_{2,s}$.

Lemma 3.2 *Suppose that (x_0, x_1, x_2, x_3) is an s -chain. Then $(x_0, x_3) \notin \Gamma_{3,3}$, $\Gamma_3 = \Gamma_{\tilde{\partial}(x_0, x_3)}$ and $k_3 \in \{1, 3\}$.*

Proof. Let $\Gamma_{1,2}(x_2) = \{x_3, x'_3, x''_3\}$. Since $p_{(2,1),(2,1)}^{(1,2)} = p_{(1,2),(1,2)}^{(2,1)} = 2$, we obtain $x'_3, x''_3 \in \Gamma_{2,1}(x_1)$, which implies $x'_3, x''_3 \in \Gamma_{3,3}(x_0)$ by $\Gamma_{1,2}\Gamma_{2,1} = \{\Gamma_{0,0}, \Gamma_{3,3}\}$. If $x_3 \in \Gamma_{3,3}(x_0)$, from Lemma 2.1 (i), then $p_{(2,s),(1,2)}^{(3,3)} = k_{2,s}k_{1,2}/k_{3,3} = 3/2$, a contradiction. Hence, $(x_0, x_3) \notin \Gamma_{3,3}$ and $\Gamma_{2,s}\Gamma_{1,2} = \{\Gamma_{3,3}, \Gamma_{\tilde{a}_3}\}$ with $\tilde{\partial}(x_0, x_3) = \tilde{a}_3$.

For any s -chain (y_0, y_1, y_2, y_3) , similarly, $\tilde{\partial}(y_0, y_3) \neq (3, 3)$. Hence, $\tilde{\partial}(y_0, y_3) = \tilde{\partial}(x_0, x_3)$ and $\Gamma_3 \subseteq \Gamma_{\tilde{a}_3}$. Conversely, for any $(z_0, z_3) \in \Gamma_{\tilde{a}_3}$, there exist two vertices z_1, z_2 such that $z_2 \in P_{(2,s),(1,2)}(z_0, z_3)$ and $z_1 \in P_{(1,2),(1,2)}(z_0, z_2)$ from $x_2 \in P_{(2,s),(1,2)}(x_0, x_3)$. Since $\Gamma_{1,2}\Gamma_{2,1} = \{\Gamma_{0,0}, \Gamma_{3,3}\}$, we get $\tilde{\partial}(z_1, z_3) \neq (2, 1)$, which implies $\tilde{\partial}(z_1, z_3) = (2, s)$ and $(z_0, z_3) \in \Gamma_3$. Then $\Gamma_3 \subseteq \Gamma_{\tilde{a}_3}$ and $\Gamma_3 = \Gamma_{\tilde{a}_3}$. By $\Gamma_{2,s}\Gamma_{1,2} = \{\Gamma_{3,3}, \Gamma_{\tilde{a}_3}\}$ and Lemma 2.1 (i), one has $k_3 = k_{\tilde{a}_3} \in \{1, 3\}$. \square

Lemma 3.3 *Let $(x_0, x_1, x_2, x_3, x_4)$ be an s -chain. If $p_{(1,2),(3,3)}^{\tilde{\partial}(x_0, x_4)} \neq 0$, then $k_3 = 1$.*

Proof. Pick a vertex $z_1 \in P_{(1,2),(3,3)}(x_0, x_4)$. In view of Lemma 2.5 (i), there exists $z_3 \in P_{(2,1),(1,2)}(z_1, x_4)$. Lemma 3.2 implies $z_1 \notin \{x_1, x_4\}$. By $p_{(1,2),(1,2)}^{(2,1)} = p_{(2,1),(2,1)}^{(1,2)} = 2$, we may assume $z_2 \in P_{(1,2),(1,2)}(z_1, z_3) \cap P_{(2,1),(2,1)}(z_3, x_4)$ and $P_{(1,2),(1,2)}(z_2, x_4) = \{z_3, w_3\}$. Since $z_3 \in P_{(2,1),(1,2)}(z_1, x_4)$, from Lemma 2.5 (i), we get $\tilde{\partial}(z_1, w_3) = (2, s)$.

Suppose $k_3 \neq 1$. By Lemma 3.2, $k_3 = 3$. It follows from Lemma 2.1 (ii) that $p_{(3,3),(1,2)}^{\tilde{a}_4} k_{\tilde{a}_4} = p_{\tilde{a}_4,(2,1)}^{(3,3)} k_{3,3}$ and $p_{\tilde{a}_3,(1,2)}^{\tilde{a}_4} k_{\tilde{a}_4} = p_{\tilde{a}_4,(2,1)}^{\tilde{a}_3} k_{\tilde{a}_3}$ with $\tilde{\partial}(x_0, x_3) = \tilde{a}_3$ and $\tilde{\partial}(x_0, x_4) = \tilde{a}_4$. Since $k_{3,3} = 6$ and $k_{\tilde{a}_3} = k_3 = 3$, one gets $p_{(3,3),(1,2)}^{\tilde{a}_4} / p_{\tilde{a}_3,(1,2)}^{\tilde{a}_4} \geq \frac{2}{3}$,

which implies $p_{\tilde{a}_3, (1,2)}^{\tilde{a}_4} = 1$ by Lemma 2.1 (iv). In view of $x_1 \neq z_1$ and Lemma 3.2, we obtain $(x_0, w_3) \notin \Gamma_3$. Hence, $\tilde{\partial}(x_0, z_2) = (2, 1)$.

Observe $(x_0, x_4) \in \Gamma_4$. By Lemma 2.5 (i) and Lemma 3.2, we obtain $\Gamma_{2,1} \cap \{(z_2, x_1), (x_3, z_2), (x_4, x_0)\} = \emptyset$, which implies $\{(z_2, x_1), (x_3, z_2), (x_4, x_0)\} \subseteq \Gamma_{2,s}$. Hence, $(x_0, x_1, x_2, x_3, x_4, z_2, x_0)$ is a closed s -chain. It follows that $\partial(x_0, x_3) = \partial(x_3, x_0) = 2$ and $s = 2$.

Note that there exists a vertex $w_1 \in P_{(1,2),(1,2)}(x_0, z_2) \setminus \{z_1\}$. By Lemma 2.5 (i) and Lemma 3.2 again, we have $\Gamma_{1,2}(x_0) = \{x_1, z_1, w_1\}$. Since $\Gamma_{1,2} \notin \Gamma_{1,2}^2$, we get z_1 or $w_1 \in P_{(1,2),(1,2)}(x_0, x_3)$. Observe $z_3 \in P_{(2,1),(1,2)}(z_1, x_4)$ and $z_2 \in P_{(1,2),(2,1)}(w_1, x_4)$, contrary to $\Gamma_{1,2}\Gamma_{2,1} = \{\Gamma_{0,0}, \Gamma_{3,3}\}$. Thus, $k_3 = 1$. \square

Lemma 3.4 *Let (x_0, x_1, \dots, x_i) be an s -chain with $4 \leq i \leq n-1$, where n is the length of an s -line. Then $\Gamma_i = \Gamma_{\tilde{\partial}(x_0, x_i)}$ and $k_i \in \{1, 3\}$.*

Proof. Use induction on i for $4 \leq i \leq n-1$. Suppose that the lemma holds for s -chain (x_0, x_1, \dots, x_j) with $0 \leq j \leq i-1$. Write $\tilde{\partial}(x_0, x_h) = \tilde{a}_h$ for $0 \leq h \leq i$.

Case 1. $p_{(1,2),(3,3)}^{\tilde{a}_4} \neq 0$.

By Lemma 3.3, $k_3 = 1$. Let $i = 3l + r$, where $1 \leq r \leq 3$.

For any $(y_0, y_i) \in \Gamma_{\tilde{a}_i}$, from $x_r \in P_{\tilde{a}_r, \tilde{a}_{3l}}(x_0, x_i)$ and the inductive hypothesis, there exists an s -chain (y_0, y_1, \dots, y_r) with $(y_r, y_i) \in \Gamma_{3l}$. Pick a vertex y_{r+1} such that $y_{r+1} \in P_{(2,s),(2,1)}(y_{r-1}, y_r)$. By $\Gamma_{3l} = \Gamma_{\tilde{a}_{3l}} \in \Gamma_{\tilde{a}_3}^l$ and Lemma 2.1 (i), we get $k_{3l} = 1$ and there exists an s -chain $(y_r, y_{r+1}, \dots, y_i)$. Then $(y_0, y_i) \in \Gamma_i$ and $\Gamma_{\tilde{a}_i} \subseteq \Gamma_i$.

Conversely, let (y_0, y_1, \dots, y_i) be an s -chain. By Lemma 2.1 (iii), we get $\Gamma_{\tilde{a}_{3l}}\Gamma_{\tilde{a}_r} = \{\Gamma_{\tilde{a}_i}\}$, which implies $\tilde{\partial}(y_0, y_i) = \tilde{a}_i$ from the inductive hypothesis. Hence, $\Gamma_i \subseteq \Gamma_{\tilde{a}_i}$ and $\Gamma_i = \Gamma_{\tilde{a}_i}$. By Lemma 2.1 (i), if $r = 1$ or 2 , then $k_i = 3$; if $r = 3$, then $k_i = 1$.

Case 2. $p_{(1,2),(3,3)}^{\tilde{a}_4} = 0$.

Pick any $(y_0, y_i) \in \Gamma_{\tilde{a}_i}$. Suppose $(y_0, y_i) \notin \Gamma_i$. By the inductive hypothesis and $x_{i-l} \in P_{\tilde{a}_{i-l}, \tilde{a}_l}(x_0, x_i)$ with $l = 1, 2, \dots, i-1$, there exist two s -chains $(y_{l,0} = y_0, y_{l,1}, \dots, y_{l,i-l})$ and $(y_{l,i-l}, y_{l,i-l+1}, \dots, y_{l,i} = y_i)$ such that $\tilde{\partial}(y_{l,i-l-1}, y_{l,i-l+1}) = (2, 1)$. Since $\Gamma_{1,2}\Gamma_{2,1} = \{\Gamma_{0,0}, \Gamma_{3,3}\}$, we get $\{(y_{1,i-3}, y_i), (y_{2,i-3}, y_i), (y_{3,i-4}, y_{3,i-1})\} \subseteq \Gamma_{3,3}$. By $\tilde{\partial}(y_{1,i-2}, y_i) = (2, 1)$, one gets $y_{1,i-2} \neq y_{2,i-2}$ and $y_{1,1} \neq y_{2,1}$. From Lemma 3.2, we obtain $y_{3,i-3} \notin \{y_{1,i-3}, y_{2,i-3}\}$ and $\Gamma_{1,2}(y_0) = \{y_{1,1}, y_{2,1}, y_{3,1}\}$. Since $p_{(1,2),(3,3)}^{\tilde{a}_4} = 0$, we get $i \geq 5$. If $k_{i-4} = 1$, then $y_{1,i-4} = y_{4,i-4}$ and $y_{1,i-3} \in P_{(1,2),(3,3)}(y_{4,i-4}, y_i)$, a contradiction; if $k_{i-4} = 3$, then $y_{4,i-4} \in \{y_{1,i-4}, y_{2,i-2}, y_{3,i-4}\}$, and $y_{3,i-1} \in P_{(3,3),(1,2)}(y_{4,i-4}, y_i)$ or $y_{h,i-3} \in P_{(1,2),(3,3)}(y_{4,i-4}, y_i)$ for some $h \in \{1, 2\}$, a contradiction. Hence, $(y_0, y_i) \in \Gamma_i$ and $\Gamma_{\tilde{a}_i} \subseteq \Gamma_i$.

Conversely, let (y_0, y_1, \dots, y_i) be an s -chain. Suppose $\tilde{\partial}(y_0, y_i) \neq \tilde{\partial}(x_0, x_i)$. Since $\Gamma_{\tilde{a}_i} \subseteq \Gamma_i$ and $x_i \in P_{\tilde{a}_i, \tilde{a}_l^*}(x_0, x_{i-l})$ with $l = 1, 2, 3$, there exists a vertex $z_{l,i} \in P_{\tilde{a}_i, \tilde{a}_l^*}(y_0, y_{i-l}) \cap \Gamma_i(y_0)$ and $z_{l,i} \neq y_i$. Assume that $z \in P_{(1,2),(1,2)}(y_{i-2}, z_{2,i})$. In view of $p_{(2,s),(2,1)}^{(1,2)} = 1$ from Lemma 2.1 (ii), we have $\tilde{\partial}(y_{i-2}, z_{1,i}) = \tilde{\partial}(y_{i-3}, z) = (2, 1)$ and $z_{1,i} \neq z_{2,i}$. By Lemma 2.5 (i), we obtain $\tilde{\partial}(y_{i-3}, z_{l,i}) = (3, 3)$ for any $l = 1, 2$. Since $z_{3,i} \in \Gamma_3(y_{i-3})$, from Lemma 3.2, one gets $z_{3,i} \notin \{y_i, z_{1,i}, z_{2,i}\}$ and $\{y_i, z_{1,i}, z_{2,i}, z_{3,i}\} \subseteq \Gamma_i(y_0)$, contrary to $|\Gamma_i(y_0)| \leq 3$. Hence, $\tilde{\partial}(y_0, y_i) = \tilde{\partial}(x_0, x_i)$ and $\Gamma_i \subseteq \Gamma_{\tilde{a}_i}$. Thus, $\Gamma_i = \Gamma_{\tilde{a}_i}$.

At last, we will prove $k_i \in \{1, 3\}$ for this case. Suppose $k_i = 2$. For any vertex x_0 , let $\Gamma_{1,2}(x_0) = \{x_1, y_1, z_1\}$ and (x_0, x_1, \dots, x_i) , $(x_0, y_1, y_2, \dots, y_i)$, $(x_0, z_1, z_2, \dots, z_i)$ be three s -chains. Without loss of generality, we may assume $x_i = y_i$ and $z_i \neq x_i$. By Lemma 2.1 (ii), we have $k_{\tilde{a}_i} p_{(1,2), \tilde{a}_{i-1}}^{\tilde{a}_i} = k_{1,2} p_{\tilde{a}_i, \tilde{a}_{i-1}}^{(1,2)}$, which implies $p_{(1,2), \tilde{a}_{i-1}}^{\tilde{a}_i} = k_{\tilde{a}_{i-1}} = 3$. From the commutativity of Γ , $\Gamma_{2,1}(x_i) = \Gamma_{2,1}(z_i) = \{x_{i-1}, y_{i-1}, z_{i-1}\}$. Since $p_{(2,s), (2,1)}^{(1,2)} = 1$, we get $\{(z_{i-2}, x_i), (y_{i-2}, z_i), (x_{i-2}, z_i)\} \subseteq \Gamma_{2,1}$ and $k_{i-2} = 3$. But $P_{\tilde{a}_{i-2}, (2,s)}(x_0, x_i) = \{x_{i-2}, y_{i-2}\}$ and $P_{\tilde{a}_{i-2}, (2,s)}(x_0, z_i) = \{z_{i-2}\}$, a contradiction. Hence, $k_i \in \{1, 3\}$. \square

Lemma 3.5 *Let n be the length of an s -line.*

- (i) *If (x_0, x_1, \dots, x_n) is an s -chain, then $x_0 = x_n$.*
- (ii) *For any s -chain (y_0, y_1, \dots, y_l) , $y_i = y_j$ if and only if $i \equiv j \pmod{n}$ with $0 \leq i, j \leq l$.*

Proof. (i) By Lemmas 3.2 and 3.4, one has $\Gamma_{n-1} = \Gamma_{2,1}$ and $\Gamma_{n-2} = \Gamma_{s,2}$. Lemma 2.1 (ii) implies $p_{(2,s), (2,1)}^{(1,2)} = 1$. Since $x_0, x_n \in P_{(2,s), (2,1)}(x_{n-2}, x_{n-1})$, we get $x_n = x_0$. (ii) is obvious by (i). \square

Lemma 3.6 *Let $L = (y_0, y_1, \dots, y_j)$ be an s -chain. Then the following hold:*

- (i) *If L is an s -line, then $\hat{L} := (y_1, y_2, \dots, y_n, y_1)$ is an s -line and $(L, \hat{L}) \in \tilde{\Gamma}_{1,2}$.*
- (ii) *For each $y'_0 \in \Gamma_{1,2}(y_0) \setminus \{y_1\}$, there exists a unique s -chain L' starting from y'_0 such that $(L, L') \in \tilde{\Gamma}_{1,2}$.*
- (iii) *If $(L, L') \in \tilde{\Gamma}_{1,2}$ and $L' \neq \hat{L}$, then there exists a unique s -chain L'' such that $(L', L'') \in \tilde{\Gamma}_{1,2}$ and $(L, L'') \in \tilde{\Gamma}_{2,s}$.*

Proof. (i) is an immediate consequence of Lemma 3.5.

We define inductively y'_i to be the unique element in $P_{(1,2), (2,1)}(y'_{i-1}, y_i)$ for $1 \leq i \leq j$, which is well-defined by Lemma 2.5 (i) and $y'_{i-1} \neq y_i$ since $p_{(1,2), (1,2)}^{(2,s)} = 1$. It remains to show that $L' := (y'_0, y'_1, \dots, y'_j)$ is an s -chain. Suppose for the contrary that $\tilde{\partial}(y'_i, y'_{i+2}) = (2, 1)$ for some $i \in \{0, 1, \dots, j-2\}$. By $\tilde{\partial}(y_i, y_{i+2}) = (2, s)$, one gets $y_i \neq y'_{i+2}$. Since $\tilde{\partial}(y_{i+1}, y'_{i+2}) = (2, 1)$ from $p_{(1,2), (1,2)}^{(2,s)} = 1$, we have $y'_i, y_{i+1} \in P_{(1,2), (2,1)}(y_i, y'_{i+2})$, contrary to Lemma 2.5 (i). This proves (ii).

Let $L' = (y'_0, y'_1, \dots, y'_j)$. Then $y'_0 \neq y_1$ and $\tilde{\partial}(y_0, y'_1) = (2, 1)$. Hence, there exists $y''_0 \in P_{(1,2), (s,2)}(y'_0, y_0) \subseteq \Gamma_{1,2}(y'_0) \setminus \{y'_1\}$ and a unique s -chain $L'' = (y''_0, y''_1, \dots, y''_j)$ such that $(L', L'') \in \tilde{\Gamma}_{1,2}$ by (ii). Since (y_0, y'_0, y''_0) is an s -chain, from (ii) and the inductive hypothesis, we obtain $(y_i, y''_i) \in \Gamma_{2,s}$ for $1 \leq i \leq j$. Thus, (iii) is valid. \square

By Lemma 3.6 (ii) and (iii), there exists an s -plane (L_0, L_1, \dots, L_n) with $L_n = L_0$. For the remainder of this section, we assume $L_j := (y(j, 0), y(j, 1), \dots, y(j, n))$ and $y(i, j+1) \neq y(i+1, j)$ for $0 \leq i, j \leq n$, where the coordinates could be read modulo n .

Lemma 3.7 For $0 \leq i, j \leq n-1$, the following hold:

- (i) $\partial(y(i, j), y(i-1, j-1)) = 1$ and $\tilde{\partial}(y(i, j), y(i-2, j-2)) = (2, s)$.
- (ii) $\{y(i+1, j-1), y(i-1, j+1), y(i+2, j+1), y(i+1, j+2), y(i-2, j-1), y(i-1, j-2)\} \subseteq \Gamma_{3,3}(y(i, j))$.

Proof. (i) Since $y(i, j-1), y(i-1, j) \in P_{(1,2),(1,2)}(y(i-1, j-1), y(i, j))$ and $y(i, j-1) \neq y(i-1, j)$, we have $\partial(y(i, j), y(i-1, j-1)) = 1$ by $p_{(1,2),(1,2)}^{(2,s)} = 1$.

Suppose $\tilde{\partial}(y(i, j), y(i-2, j-2)) = (2, 1)$ for some i, j . Since $y(i-1, j-2), y(i-2, j-1), y(i, j) \in P_{(1,2),(1,2)}(y(i-2, j-2), y(i-1, j-1))$ and $y(i-1, j-2) \neq y(i-2, j-1)$, one gets $y(i, j) \in \{y(i-1, j-2), y(i-2, j-1)\}$ by $p_{(1,2),(1,2)}^{(2,1)} = 2$. In view of the symmetry, we may assume $y(i, j) = y(i-1, j-2)$. It follows $(2, s) = \partial(y(i, j-2), y(i, j)) = \partial(y(i, j-2), y(i-1, j-2)) = (2, 1)$, a contradiction.

(ii) Since $k_{1,2} = 3$ and $\Gamma_{1,2}\Gamma_{2,1} = \{\Gamma_{0,0}, \Gamma_{3,3}\}$, we have $\Gamma_{1,2}(y(h, l)) \setminus \{y(i, j)\} \subseteq \Gamma_{3,3}(y(i, j))$ for any $(h, l) \in \{(i-1, j), (i, j-1), (i+1, j+1)\}$. Hence, (ii) is valid. \square

For any vertex $y(i, j)$, there are three distinct s -lines beginning from $y(i, j)$. They are $(y(i, j), y(i+1, j), \dots, y(i, j))$, $(y(i, j), y(i, j+1), \dots, y(i, j))$ and $(y(i, j), y(i-1, j-1), \dots, y(i, j))$. We consider the cardinal of $\{y(i, j) \mid 0 \leq i, j \leq n-1\}$.

Proposition 3.8 If $|\{y(i, j) \mid 0 \leq i, j \leq n-1\}| = n^2$, then Γ is isomorphic to one of the digraphs in Theorem 1.1 (vii).

Proof. Since $s \neq 1$, we obtain $n \geq 4$. If $3 \mid n$, then $\tilde{\partial}(y(0, 0), y(1+n/3, 0)) = \tilde{\partial}(y(0, 0), y(1, n/3)) = (1+n/3, -1+2n/3)$, contrary to $y(1, n/3) \notin \Gamma_{1+n/3}(y(0, 0))$. Thus, the desired result holds. \square

Finally, we consider the case that $|\{y(i, j) \mid 0 \leq i, j \leq n-1\}| \neq n^2$.

Lemma 3.9 For $1 \leq l \leq n-1$, $k_l = 1$ if and only if $3l \equiv 0 \pmod{n}$ and $y(a, b) = y(a+l, b+2l) = y(a+2l, b+l)$ for any $a, b \in \{0, 1, \dots, n-1\}$.

Proof. " \implies " Since $k_l = 1$, we have $y(a+l, b) = y(a, b+l)$, which implies that $y(a+l, b) = y(a-2l, b)$ by $|\Gamma_l(y(a-l, b+l))| = 1$. Hence, $3l \equiv 0 \pmod{n}$. In view of $|\Gamma_l(y(a, b+2l))| = 1$, one gets $y(a, b) = y(a+l, b+2l) = y(a+2l, b+l)$.

" \impliedby " Since $|\Gamma_l(y(a, b-l))| = 1$, we obtain $k_l = 1$. \square

Lemma 3.10 If $y(a, b) = y(a+i, b+j)$ for some $a, b \in \{0, 1, \dots, n-1\}$ and $i, j \in \{1, 2, \dots, n-1\}$, then $k_i = k_j$.

Proof. By Lemmas 3.2 and 3.4, we only need to prove that $k_i = 1$ if and only if $k_j = 1$. For any $l \in \{i, j\}$, from $|\Gamma_l(y(a, b-l))| \in \{1, 3\}$ and Lemma 3.9, $k_l = 1$ if and only if $y(a, b) = y(a+l, b+2l) = y(a+2l, b+l) = y(a+i, b+j)$ and $3l \equiv 0 \pmod{n}$, if and only if $j \equiv 2i \pmod{n}$ and $i \equiv 2j \pmod{n}$. This completes the proof of the lemma. \square

If $\Gamma_l = \Gamma_{\tilde{f}}$ for some $\tilde{f} \in \{\tilde{i}, \tilde{j}, \tilde{h}\}$, then we may replace \tilde{f} with l in the intersection number $p_{i,j}^{\tilde{h}}$ and the set $P_{i,j}(x, y)$ for $x, y \in V\Gamma$.

Lemma 3.11 *If $y(0, 0) = y(i, j)$ with $1 \leq i, j \leq n - 1$ and $i \neq j$, then $k_i = 1$.*

Proof. Suppose for the contrary that $k_i = 3$.

Step 1 Show that $n \nmid (h + l)$ and $y(0, 0) = y(h - l, h) = y(2h, 2l)$ if $k_h = 3$ and $y(0, 0) = y(h, l)$ with $n \nmid h$ and $n \nmid (h - l)$.

By Lemma 3.5, we may assume that $1 \leq h, l \leq n - 1$. Since $\Gamma_h(y(0, l)) = \{y(h, l), y(0, h + l), y(-h, l - h)\}$, one gets $y(0, 0) = y(h, l) \neq y(0, h + l)$, which implies $n \nmid (h + l)$. By $y(h, l) \in P_{0, n-l}(y(0, 0), y(h, 0))$, we get $\emptyset \neq P_{0, n-l}(y(h, l), y(h, h + l)) \subseteq \{y(h + l, h + l), y(h, h + 2l), y(h - l, h)\}$ and $\emptyset \neq P_{0, n-l}(y(h, l), y(2h, l)) \subseteq \{y(2h, 2l), y(2h + l, l), y(2h - l, 0)\}$. Since $n \nmid (h + l)$ and $y(h, l) = y(0, 0)$, one has $y(0, 0) = y(h - l, h)$ and $y(0, 0) \in \{y(2h, 2l), y(2h - l, 0)\}$. By $\Gamma_h(y(h - l, 0)) = \{y(2h - l, 0), y(h - l, h), y(-l, -h)\}$, we obtain $y(0, 0) = y(2h, 2l)$.

Step 2 Show that $\Gamma_{i+j} \neq \Gamma_{2n-i-j}$.

Suppose $\Gamma_{i+j} = \Gamma_{2n-i-j}$. Since $p_{0, 2n-i-j}^{i+j} = p_{0, i+j}^{i+j} \neq 0$ and $P_{0, 2n-i-j}(y(0, 0), y(i + j, 0)) \subseteq \{y(i + j, i + j), y(2i + 2j, 0), y(0, -i - j)\}$, we have $y(2i + 2j, 0) = y(0, 0)$ and $i + j \equiv n/2 \pmod{n}$ from Step 1. By Lemma 3.10, we get $k_{i-j} = k_i = k_j = 3$, which implies $y(2i, 2j) = y(i - j, i) = y(i - j - i, i - j) = y(-2j, 2i - 2j) = y(2i, -4j)$. Hence, $6j \equiv 0 \pmod{n}$. Since $i + j \equiv n/2 \pmod{n}$ and $j \neq i$, we obtain $(i, j) \in \{(n/3, n/6), (n/6, n/3), (5n/6, 2n/3), (2n/3, 5n/6)\}$. Then $y(i, j) = y(0, 0) = y(2i, 2j) = y(n/3, 2n/3)$ or $y(2n/3, n/3)$. Since i or $j \in \{n/3, 2n/3\}$, by Lemma 3.10 and Step 1, one has $y(0, 0) = y(i, j) = y(4i, 4j) = y(2n/3, n/3)$ or $y(n/3, 2n/3)$, a contradiction.

Step 3 Show that $p_{(3,3),(3,3)}^3 = p_{(3,3),(3,3)}^{n-3} = 2$.

Suppose $k_3 = 1$. By Lemma 3.9, we have $n = 9$ and $y(0, 0) = y(3, 6) = y(6, 3)$, which imply $i \in \{1, 2, 4, 5, 7, 8\}$ since $k_i = 3$. Then $y(0, 0) = y(1, l)$ for some l by repeating using $k_{2i} = 3$ and $y(0, 0) = y(2i, 2j)$ from Step 1. In view of Lemma 3.7 (ii), one gets $y(1, 2), y(1, 8) \in \Gamma_{3,3}(y(0, 0))$ and $y(1, 5) \in \Gamma_{3,3}(y(3, 6))$. Since $\tilde{\partial}(y(0, 0), y(1, 1)) = (2, 1)$, we obtain $l \in \{4, 7\}$. By Step 1, one has $y(1 - l, 1) = y(0, 0) = y(6, 3) = y(3, 6)$, contrary to $n = 9$. Hence, $k_3 = 3$.

Suppose $\Gamma_3 = \Gamma_{n-3}$. By Lemma 3.2, we have $\Gamma_3 = \Gamma_{2,2}$ and $s = 2$. Pick a circuit (x, y, z, w) such that $\tilde{\partial}(x, z) = (2, 2)$. Since $\Gamma_{1,2} \notin \Gamma_{1,2}^2$, one gets $\tilde{\partial}(y, w) = (2, 2)$. Note that (x, y, z, w, x) is a closed 2-chain, contrary to $\Gamma_3 = \Gamma_{2,2}$. Thus, $\Gamma_3 \neq \Gamma_{n-3}$.

By Lemma 2.1 (ii) and (iv), we get $k_3 p_{(3,3),(3,3)}^3 = k_{n-3} p_{(3,3),(3,3)}^{n-3} = k_{3,3} p_{(3,3),n-3}^{(3,3)} = k_{3,3} p_{(3,3),3}^{(3,3)}$ and $k_{3,3} \geq p_{(3,3),(3,3)}^{(3,3)} + p_{(3,3),3}^{(3,3)} + p_{(3,3),n-3}^{(3,3)} + p_{(3,3),(0,0)}^{(3,3)}$. Since $y(i - j + 1, 2 - j), y(i - j + 1, -j - 1) \in P_{(3,3),(3,3)}(y(i - j + 2, 1 - j), y(i - j, -j))$ and $y(0, 0) \in P_{(3,3),(3,3)}(y(i - 2, j - 1), y(i + 1, j - 1))$ from Lemma 3.7 (ii), one has $p_{(3,3),(3,3)}^{(3,3)} \geq 2$ and $p_{(3,3),(3,3)}^3 = p_{(3,3),(3,3)}^{n-3} \neq 0$, which imply $p_{(3,3),3}^{(3,3)} = 1$ and $p_{(3,3),(3,3)}^3 = p_{(3,3),(3,3)}^{n-3} = 2$.

Based on the above discussion, we reach a contradiction as follows.

For any $(h, l) \in \{(i+1, j-1), (i+1, j+2), (i-2, j-1)\}$, in view of Lemma 3.7 (ii), we have $y(h, l) \in \Gamma_{3,3}(y(0, 0)) = \Gamma_{3,3}(y(i, j))$ and $y(h, 0) \in P_{n+h,l}(y(0, 0), y(h, l))$. Since $k_{3,3} = 6$ and $k_{3,3}p_{n+h,l}^{(3,3)} = k_{n+h}p_{(3,3),2n-l}^{n+h}$ from Lemma 2.1 (ii), we get $p_{(3,3),2n-l}^{n+h} = 2$ by Lemmas 3.2 and 3.4. Hence, $|\Gamma_{3,3}(y(0, 0)) \cap \Gamma_l(y(h, 0))| = 2$. Since $\Gamma_l(y(h, 0)) = \{y(h, l), y(h+l, 0), y(h-l, -l)\}$, we obtain $\{y(h+l, 0), y(h-l, -l)\} \cap \Gamma_{3,3}(y(0, 0)) \neq \emptyset$.

By Step 2, we have $y(i+j, 0) \notin \Gamma_{3,3}(y(0, 0))$ and $y(i-j+2, 1-j) \in \Gamma_{3,3}(y(0, 0))$. Suppose $y(i-j-1, a-j) \in \Gamma_{3,3}(y(0, 0))$ for some $a \in \{1, -2\}$. From Lemma 3.7 (ii), one gets $y(0, 0), y(i-j, -j), y(i-j+1, 1-j+a) \in P_{(3,3),(3,3)}(y(i-j+2, 1-j), y(i-j-1, a-j))$. By Step 1, we have $y(0, 0) = y(i-j, i)$ and $y(0, 0) \in \Gamma_{i+j}(y(i-j, -j))$, which imply $y(0, 0) \neq y(i-j, -j)$. In view of $y(i-j+1, 1-j+a) \in \Gamma_{3,3}(y(i-j, -j))$ and Step 3, we obtain $y(0, 0) = y(i-j+1, 1-j+a)$, which implies $\partial(y(0, 0), y(i-j, -j)) = \partial(y(i-j+1, 1-j+a), y(i-j, -j)) = (3, 3)$, contrary to Step 2. Hence, $\{y(i+j+3, 0), y(i+j-3, 0)\} \subseteq \Gamma_{3,3}(y(0, 0))$.

For any $(h', l') \in \{(i+1, j+2), (i-2, j-1)\}$, since $p_{0,2n-h'-l'}^{h'+l'} = p_{0,(3,3)}^{(3,3)} \neq 0$, we have $y(0, 0) \in \Gamma_{h'+l'}(y(h'+l', 0))$. By $n \nmid (h'+l')$, $y(0, 0) = y(2h'+2l', 0)$. Therefore, $y(0, 0) = y(2i+2j+6, 0) = y(2i+2j-6, 0)$. It follows that $12 \equiv 0 \pmod{n}$. Since $n \geq 4$, $n \in \{4, 6, 12\}$. By $n \nmid (i+j)$, we get $i+j \equiv 3 \pmod{n}$ or $i+j \equiv n-3 \pmod{n}$, contrary to $\{y(i+j+3, 0), y(i+j-3, 0)\} \subseteq \Gamma_{3,3}(y(0, 0))$. Thus, $k_i = 1$. \square

Proposition 3.12 *If $|\{y(i, j) \mid 0 \leq i, j \leq n-1\}| \neq n^2$, then $3 \mid n$, $n \geq 9$ and $\Gamma \simeq \text{Cay}(\mathbb{Z}_{n/3} \times \mathbb{Z}_n, \{(0, 1), (1, 1), (-1, -2)\})$.*

Proof. For some $a, b \in \{0, 1, \dots, n-1\}$, suppose $y(a, b) = y(a+i, b+j)$ with $i, j \in \{1, 2, \dots, n-1\}$ and $i \neq j$. Since $y(a+i, b+j) \in P_{0,n-j}(y(a, b), y(a+i, b))$ and $y(a+i, b+j) \in P_{0,n+i-j}(y(a, b), y(a+i, b+i))$, one gets $y(0, 0) \in P_{0,n-j}(y(0, 0), y(i, 0)) \subseteq \{y(i, j), y(i+j, 0), y(i-j, -j)\}$ and $y(0, 0) \in P_{0,n+i-j}(y(0, 0), y(-i, 0)) \subseteq \{y(j-2i, 0), y(-i, j-i), y(-j, i-j)\}$. If $y(0, 0) \in \{y(i, j), y(i-j, -j), y(-i, j-i), y(-j, i-j)\}$, from Lemmas 3.10 and 3.11, then $k_i = k_{n-i} = 1$ or $k_j = k_{n-j} = 1$; if $y(0, 0) = y(i+j, 0) = y(j-2i, 0)$, then $3i \equiv 0 \pmod{n}$ and $j \equiv 2i \pmod{n}$, which imply that $y(a, b) = y(a+i, a+j) = y(a+i, b+2i)$. By Lemma 3.10 or $|\Gamma_i(a+2i, b)| \in \{1, 3\}$, we have $k_i = 1$, which implies $3 \mid n$, $k_{n/3} = 1$ and $(i, j) \in \{(n/3, 2n/3), (2n/3, n/3)\}$.

Since $k_1 = k_2 = 3$, $n \geq 9$. By $k_{n/3} = 1$, one gets $|\Gamma_{n/3}(y(c+2n/3, d))| = 1$ for $0 \leq c, d \leq n-1$, which implies $y(c, d) = y(c+n/3, d+2n/3) = y(c+2n/3, d+n/3)$ and $|\{y(i, j) \mid 0 \leq i, j \leq n-1\}| = n^2/3$. Thus, $\Gamma \simeq \Gamma'_n$ for $n \geq 9$, where $\Gamma'_n = \text{Cay}(\mathbb{Z}_n \times \mathbb{Z}_n / \{(0, 0), (n/3, 2n/3), (2n/3, n/3)\}, \{(0, 1), (1, 0), (-1, -1)\})$ and $(a, b) = (a, b) + \{(0, 0), (n/3, 2n/3), (2n/3, n/3)\}$ for $0 \leq a, b \leq n-1$. Let σ be the mapping from Γ'_n to $\text{Cay}(\mathbb{Z}_{n/3} \times \mathbb{Z}_n, \{(0, 1), (1, 1), (-1, -2)\})$ such that $\sigma((a, b)) = (a, a+b)$. Routinely, σ is a desired isomorphism. Thus, the desired result holds. \square

4 $p_{(1,g-1),(1,g-1)}^{(2,g-2)} = 1$ and $\Gamma_{1,g-1} \notin \Gamma_{1,g-1}^2$

In this case, based on Lemma 2.7, we prove our main result according to two separate assumptions. And we use $(g-2)$ -chain to deal with the proof under the second assumption.

Lemma 4.1 Suppose $k_{2,s} = 1$. Then $\Gamma_{2,g-2} \notin \Gamma_{1,g-1}^2$.

Proof. The proof is rather long, and we shall prove it in Section 5.

Proposition 4.2 Suppose $k_{2,s} = 1$. Then Γ is isomorphic to the digraph in Theorem 1.1 (iv).

Proof. Pick a circuit $(x_{0,0}, x_{1,0}, \dots, x_{g-1,0})$, where the first subscription of x could be read modulo g . By Lemma 2.7 (ii), we have $p_{(1,g-1),(1,g-1)}^{(2,s)} = 3$. Since $p_{(2,s),(g-1,1)}^{(1,g-1)} = 1$ from Lemma 2.1 (ii), there exists a unique vertex $x_{i,1} \in P_{(2,s),(g-1,1)}(x_{i-1,0}, x_{i,0})$. Then $(x_{0,1}, x_{1,1}, \dots, x_{g-1,1})$ is a minimal circuit.

Since $p_{(1,g-1),(g-1,1)}^{(2,g-2)} \neq 0$ from Lemma 4.1, there exists a vertex x_0 such that $x_0 \in P_{(1,g-1),(g-1,1)}(x_{0,0}, x_{2,0})$. The fact $p_{(1,g-1),(1,g-1)}^{(2,s)} = 3$ and $(x_{0,0}, x_{1,1}), (x_{2,0}, x_{3,1}) \in \Gamma_{2,s}$ imply that $\Gamma_{1,g-1}(x_{0,0}) = \{x_{1,0}, x_{0,1}, x_0\}$ and $\Gamma_{1,g-1}(x_{2,0}) = \{x_{2,1}, x_{3,0}, x_0\}$. Similarly, there exists a vertex $x_1 \in P_{(1,g-1),(g-1,1)}(x_{0,1}, x_{2,1})$ such that $(x_{1,1}, x_1) \notin \Gamma_{2,s}$. By $\Gamma_{1,g-1} \notin \Gamma_{1,g-1}^2$, we obtain $\partial(x_{2,0}, x_{0,0}) \neq 1$ and $g \geq 4$.

Suppose $A_{1,g-1}^2 = A_{2,g-2} + A_{2,l} + 3A_{2,s}$ with $g-2 < l < s$. Then $(x_{1,0}, x_0) \in \Gamma_{2,l}$. Since $x_{0,0} \in P_{(g-1,1),(1,g-1)}(x_{1,0}, x_0)$, from the commutativity of Γ , we have $\Gamma_{2,l}, \Gamma_{l,2}, \Gamma_{2,g-2}, \Gamma_{g-2,2} \in \Gamma_{1,g-1}\Gamma_{g-1,1}$. By Lemma 2.1 (iii), one gets $(2, l) \in \{(g-2, 2), (l, 2)\}$, contrary to $g-2 < l < 2$. It follows from Lemma 2.7 (ii) that $A_{1,g-1}^2 = A_{2,g-2} + 3A_{2,s}$. In view of Lemma 2.1 (i), we obtain $k_{2,g-2} = 6$ and $|\Gamma_{1,g-1}\Gamma_{g-1,1}| = 2$. Since $p_{(1,g-1),(g-1,1)}^{(g-2,2)} \neq 0$, one has $g = 4$.

Pick a vertex x_2 such that $\Gamma_{1,3}(x_{2,1}) = \{x_{3,1}, x_1, x_2\}$. Since $x_1, x_{3,1} \in \Gamma_{2,2}(x_{1,1})$, we have $(x_{1,1}, x_2) \in \Gamma_{2,s}$ and $(x_{2,0}, x_2) \in \Gamma_{2,2}$. The fact that $p_{(1,3),(3,1)}^{(2,2)} \neq 0$ implies x_0 or $x_{3,0} \in P_{(1,3),(3,1)}(x_{2,0}, x_2)$. By $p_{(1,2),(1,2)}^{(2,s)} = 3$, one gets $x_0 \in P_{(1,2),(1,2)}(x_{0,0}, x_{1,1})$. From $\partial(x_{2,2}, x_{1,1}) = s > 2$, (x_2, x_0) is not an arc and $\partial(x_2, x_{3,0}) = 1$. Since $(x_2, x_{3,0}, x_{0,0}, x_{0,1}, x_{1,1})$ is a path, we have $s \leq 4$. If $s = 3$, by Lemma 2.1 (iii), then $\Gamma_{2,3}\Gamma_{2,3} = \{\Gamma_{3,1}\}$ or $\Gamma_{2,3}\Gamma_{2,2} = \{\Gamma_{3,1}\}$, contrary to Lemma 2.1 (i). Thus, $s = 4$.

Since $\Gamma_{1,3}^2 = \{\Gamma_{2,2}, \Gamma_{2,4}\}$, by Lemma 2.1 (iii), we have $\tilde{\partial}(x_{0,0}, x_{2,1}) = (3, 3)$ and $\Gamma_{2,4}\Gamma_{1,3} = \{\Gamma_{3,3}\}$, which imply $k_{3,3} = 3$ from Lemma 2.1 (i). In view of $\tilde{\partial}(x_{0,0}, x_{3,0}) = (3, 1)$ and $\partial(x_{0,0}, x_0) = 1$, one gets $\Gamma_{2,2}\Gamma_{1,3} = \{\Gamma_{3,1}, \Gamma_{1,3}, \Gamma_{3,3}\}$ and $\Gamma_{1,3}^3 = \{\Gamma_{1,3}, \Gamma_{3,1}, \Gamma_{3,3}\}$.

Note that $x_{1,1} \in P_{(2,4),(2,4)}(x_{0,0}, x_2)$ and $\partial(x_2, x_{0,0}) = 2$. By Lemma 2.1 (i), $\tilde{\partial}(x_{0,0}, x_2) = (4, 2)$. The fact $x_1 \neq x_{1,1}$ implies $(x_{0,0}, x_1) \in \Gamma_{2,2}$. Since $x_{0,1} \in P_{(1,3),(3,1)}(x_{3,1}, x_{0,0})$, one gets $\tilde{\partial}(x_{0,0}, x_{3,1}) = (2, 2)$. By $\Gamma_{1,3}(x_{2,1}) = \{x_1, x_2, x_{2,2}\}$, one has $\Gamma_{3,3}\Gamma_{1,3} = \{\Gamma_{2,2}, \Gamma_{4,2}\}$ and $\Gamma_{1,3}^4 = \{\Gamma_{2,2}, \Gamma_{4,2}, \Gamma_{0,0}, \Gamma_{2,4}\}$. From Lemma 2.1 (iii), $\Gamma_{4,2}\Gamma_{1,3} = \{\Gamma_{3,1}\}$. Then $\Gamma_{1,3}^{2n-1} = \{\Gamma_{1,3}, \Gamma_{3,1}, \Gamma_{3,3}\}$ and $\Gamma_{1,3}^{2n} = \{\Gamma_{2,2}, \Gamma_{4,2}, \Gamma_{0,0}, \Gamma_{2,4}\}$ for $n \geq 2$. Therefore, $\tilde{\partial}(\Gamma) = \{(0, 0), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2)\}$ and $|V\Gamma| = 18$. By [3], the desired result holds. \square

In the following of this section, from Lemma 2.7, we may assume $k_{2,s} = 3$ and $A_{1,g-1}^2 = A_{2,g-2} + 2A_{2,s}$. In view of Lemma 2.1 (i), we have $k_{2,g-2} = 3$, which implies $p_{(2,g-2),(g-1,1)}^{(1,g-1)} = 1$ and $p_{(2,s),(g-1,1)}^{(1,g-1)} = 2$ by Lemma 2.1 (ii).

Note that any $(g-2)$ -line is a minimal circuit and the length of a $(g-2)$ -line is g . As an application of $(g-2)$ -chain, we give a construction of the digraph Γ .

Lemma 4.3 *Let x be a vertex and $\Gamma_{1,g-1}(x) = \{y_0, y_1, y_2\}$.*

- (i) *If there exist two vertices z and w such that $z \in P_{(2,s),(g-1,1)}(x, y_0)$ and $w \in P_{(1,g-1),(g-1,1)}(y_0, y_2)$ (See Figure 1), then $\partial(y_1, z) = 1$.*
- (ii) *If there exist three vertices z_0, z_1, z_2 such that $\Gamma_{g-1,1}(x) = \{z_0, z_1, z_2\}$ and $\tilde{\partial}(z_0, y_0) = (2, g-2)$, then $\{(z_0, y_1), (z_0, y_2), (z_1, y_0), (z_2, y_0)\} \subseteq \Gamma_{2,s}$.*

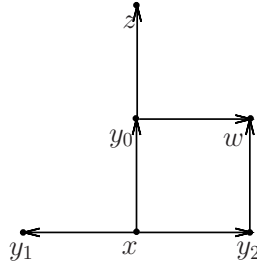


Figure 1: Lemma 4.3 (i).

Proof. (i) By Lemma 2.7 (iii), we get $\partial(y_2, z) \neq 1$, which implies that $y_1 \in P_{(1,g-1),(1,g-1)}(x, z)$ since $p_{(1,g-1),(1,g-1)}^{(2,s)} = 2$.

(ii) Since $p_{(g-1,1),(2,s)}^{(1,g-1)} = p_{(2,s),(g-1,1)}^{(1,g-1)} = 2$, one has $z_1, z_2 \in P_{(g-1,1),(2,s)}(x, y_0)$ and $y_1, y_2 \in P_{(2,s),(g-1,1)}(z_0, x)$. \square

Lemma 4.4 *Let $L = (y_0, y_1, \dots, y_{g-1}, y_g)$ be a $(g-2)$ -line. Then the following hold:*

- (i) $\hat{L} := (y_1, y_2, \dots, y_g, y_1)$ is a $(g-2)$ -line and $(L, \hat{L}) \in \tilde{\Gamma}_{1,g-1}$.
- (ii) For each $y'_0 \in \Gamma_{1,g-1}(y_0) \setminus \{y_1\}$, there exists a unique $(g-2)$ -line $L' := (y'_0, y'_1, \dots, y'_g)$ such that $(L, L') \in \tilde{\Gamma}_{1,g-1}$.
- (iii) There exists a $(g-2)$ -line L_1 such that $(L', L_1) \in \tilde{\Gamma}_{1,g-1}$ and $(L, L_1) \in \tilde{\Gamma}_{2,g-2}$.

Proof. (i) is obvious.

(ii) Let $\Gamma_{1,g-1}(y_0) = \{y_1, y'_0, y''_0\}$. Observe $y_0 \in P_{(g-1,1),(1,g-1)}(y'_0, y_1)$ and $y_0 \in P_{(g-1,1),(1,g-1)}(y''_0, y_1)$. By Lemma 2.7 (iii) and the commutativity of Γ , there only exist two distinct vertices $y'_1 \in P_{(1,g-1),(g-1,1)}(y'_0, y_1)$ and $y''_1 \in P_{(1,g-1),(g-1,1)}(y''_0, y_1)$ from $A_{1,g-1}^2 = A_{2,g-2} + 2A_{2,s}$. Pick two $(g-2)$ -lines $L' := (y'_0, y'_1, \dots, y'_{g-1}, y'_g)$ and $(y''_0, y''_1, \dots, y''_{g-1}, y''_g)$. It suffices to show that $(L, L') \in \tilde{\Gamma}_{1,g-1}$. By induction, we only need to prove $\partial(y_2, y'_2) = 1$.

Let $z_0, z_1, z_2, z_3, z_4, z_5$ be vertices such that $\Gamma_{1,g-1}(y'_1) = \{z_1, z_5, y'_2\}$, $\Gamma_{1,g-1}(y''_1) = \{z_0, z_2, y''_2\}$ and $\partial(y'_0, z_4) = \partial(y''_1, z_3) = 1$. Since $p_{(2,g-2),(g-1,1)}^{(1,g-1)} = 1$, we may assume $\tilde{\partial}(y_i, z_i) = (2, s)$, $\tilde{\partial}(y_0, z_j) = \tilde{\partial}(y_1, z_{i'}) = (2, g-2)$ for $i = 0, 1, j = 2, 4$ and $i' = 3, 5$.

Suppose $\partial(y_2, y'_2) \neq 1$. Since $p_{(1,g-1),(1,g-1)}^{(2,s)} = 2$ and $\tilde{\partial}(y_1, y'_2) = (2, s)$ from Lemma 4.3 (ii), we have $\partial(y'_1, y'_2) = 1$. In view of Lemma 4.3 (i) and $\tilde{\partial}(y_i, z_i) = (2, s)$ for $i = 0, 1$, one gets $\partial(y'_0, z_0) = \partial(y_2, z_1) = 1$. Observe $y'_1, y''_1 \in P_{(2,s),(1,g-1)}(y_0, y'_2)$ and $y'_0 \in P_{(1,g-1),(2,g-2)}(y_0, y'_2)$. By the commutativity of Γ , we obtain $(y'_0, y'_2) \in \Gamma_{2,s}$ and $y'_2 \neq y''_2$. From Lemma 4.3 (i) and (ii), one has $\tilde{\partial}(y_1, y''_2) = (2, s)$ and $\partial(y_2, y''_2) = 1$. In view of $y'_1 \in P_{(1,g-1),(g-1,1)}(y''_0, y_1)$ and Lemma 2.7 (iii), we get $y'_1 \neq z_0$. Since $p_{(1,g-1),(1,g-1)}^{(2,s)} = 2$ and $p_{(1,g-1),(1,g-1)}^{(2,g-2)} = 1$, one obtains $z_2 \in P_{(1,g-1),(1,g-1)}(y''_0, y'_2)$. By $y'_0 \in P_{(g-1,1),(1,g-1)}(z_0, y'_1)$, we have z_5 or $z_1 \in \Gamma_{1,g-1}(z_0)$.

Case 1. $\partial(z_0, z_1) = 1$.

By Lemma 4.3 (ii), $y'_0, y_1 \in P_{(1,g-1),(2,s)}(y_0, z_1)$. Since $y_2 \in P_{(2,g-2),(1,g-1)}(y_0, z_1)$, from the commutativity of Γ , we get $\tilde{\partial}(y''_0, z_1) = (2, g-2)$. By $p_{(1,g-1),(1,g-1)}^{(2,g-2)} = 1$ and $z_0, y''_1 \in \Gamma_{2,s}(y_0)$, one has $|\{z_1, y''_2, y_0\}| = 3$. If $g = 3$, then $\tilde{\partial}(y_2, y''_0) = (2, s)$, which implies $y''_2, z_1, y_0 \in P_{(1,g-1),(1,g-1)}(y_2, y''_0)$, a contradiction. Hence, $g > 3$.

By Lemma 4.3 (ii), we obtain $y'_1, z_0 \in \Gamma_{2,s}(y_0)$, which implies that $\Gamma_{g-1,1}(z_1) = \{z_0, y_2, y'_1\}$ since $y_2 \in \Gamma_{2,g-2}(y_0)$. Observe that $\tilde{\partial}(y''_{g-1}, z_0) = (2, s)$ and $y''_0 \in P_{(1,g-1),(2,g-2)}(y''_{g-1,1}, z_1)$. In view of the commutativity of Γ , we get y'_1 or $y_2 \in \Gamma_{2,g-2}(y''_{g-1})$. By Lemma 4.3 (ii) again, $\tilde{\partial}(y_2, y''_3) = (2, s)$. Since $g > 3$, we have $\tilde{\partial}(y''_{g-1}, y'_1) = (2, g-2)$. Then there exists a vertex $w \in P_{(1,g-1),(1,g-1)}(y''_{g-1}, y'_1)$. By $p_{(2,g-2),(g-1,1)}^{(1,g-1)} = 1$ and $w \notin \{y'_0, y_1\}$ from Lemma 2.7 (iii), one has $\tilde{\partial}(w, z_1) = (2, g-2)$ and $\tilde{\partial}(y''_{g-1}, z_1) = (3, g-3)$, contrary to $\tilde{\partial}(y''_{g-1}, z_0) = (2, s)$.

Case 2. $\partial(z_0, z_5) = 1$.

By Lemma 4.3 (ii), we have $z_0, y'_1 \in P_{(2,s),(1,g-1)}(y_0, z_5)$, which implies $y''_0 \in P_{(1,g-1),(2,s)}(y_0, z_5)$ from the commutativity of Γ and $y_1 \in P_{(1,g-1),(2,g-2)}(y_0, z_5)$. Since $p_{(1,g-1),(1,g-1)}^{(2,g-2)} = 1$ and $y'_1 \in P_{(1,g-1),(1,g-1)}(y_1, z_5)$, one gets $\partial(y''_1, z_5) \neq 1$. In view of $p_{(1,g-1),(1,g-1)}^{(2,s)} = 2$, we obtain $z_2 \in P_{(1,g-1),(1,g-1)}(y''_0, z_5)$, which implies $z_2, y'_1 \in P_{(g-1,1),(1,g-1)}(z_5, y'_2)$, contrary to Lemma 2.7 (iii).

Hence, $\partial(y_2, y'_2) = 1$. This proves (ii).

(iii) Since $p_{(2,g-2),(g-1,1)}^{(1,g-1)} = 1$, there exists a vertex $w_0 \in P_{(2,g-2),(g-1,1)}(y_0, y'_0)$ and a unique $(g-2)$ -line $L_1 = (w_0, w_1, \dots, w_{g-1}, w_0)$ such that $(L', L_1) \in \tilde{\Gamma}_{1,g-1}$ by (ii). It suffices to show that $(y_i, w_i) \in \Gamma_{2,g-2}$ for $1 \leq i \leq g-1$. By induction, we only need to prove $\tilde{\partial}(y_1, w_1) = (2, g-2)$.

Suppose $\partial(y_1, w_1) = (2, s)$. Assume that $\Gamma_{1,g-1}(y_0) = \{y_1, y'_0, y''_0\}$. By (ii), there exists a unique $(g-2)$ -line $L'' = (y''_0, y''_1, \dots, y''_{g-1}, y''_0)$ such that $(L, L'') \in \tilde{\Gamma}_{1,g-1}$. Since $p_{(1,g-1),(1,g-1)}^{(2,g-2)} = 1$, we have $\tilde{\partial}(y_0, y'_1) = (2, s)$ and $w_0 \neq y'_1$, which imply $\tilde{\partial}(y'_0, w_1) = (2, s)$. From $y'_2 \in \Gamma_{2,g-2}(y'_0)$, $w_1 \neq y'_2$. By Lemma 4.3 (i) and $\tilde{\partial}(y_1, w_1) = (2, s)$, one gets $\partial(y''_1, w_1) = 1$. The fact that $y'_1, y''_1 \in \Gamma_{2,s}(y_0)$ and $p_{(1,g-1),(1,g-1)}^{(2,s)} = 2$ imply $y'_1 \neq y''_1$. Since $y_2 \in \Gamma_{2,g-2}(y_0)$, $\Gamma_{1,g-1}(y_1) = \{y'_1, y''_1, y_2\}$. By $y''_1, y_2 \in P_{(1,g-1),(1,g-1)}(y_1, y''_2)$, one has $w_1 \neq y''_2$, which implies $\tilde{\partial}(y'_0, w_1) = (2, s)$ from Lemma 4.3 (ii). But $y''_0, y_1, y'_0 \in P_{(1,g-1),(2,s)}(y_0, w_1)$ and $w_0 \in P_{(2,g-2),(1,g-1)}(y_0, w_1)$, contrary to the commutativity of Γ . \square

For each $\tilde{h} \in \tilde{\partial}(\Gamma)$, we define a relation $\bar{\Gamma}_{\tilde{h}}$ on the set of chains of $(g-2)$ -lines as follows. For any two chains of $(g-2)$ -lines $P = (L_0, L_1, \dots, L_j)$ and $P' = (L'_0, L'_1, \dots, L'_j)$, $(P, P') \in \bar{\Gamma}_{\tilde{h}}$ if and only if $(L_i, L'_i) \in \tilde{\Gamma}_{\tilde{h}}$ for $0 \leq i \leq j$.

Lemma 4.5 *Suppose that $P = (L_0, L_1, \dots, L_{g-1}, L_g)$ is a $(g-2)$ -plane with $L_j = (y_{j,0}, y_{j,1}, \dots, y_{j,g})$ and $y_{h,l+1} \neq y_{h+1,l}$ for $0 \leq j \leq g$ and $0 \leq h, l \leq g-1$. The following hold:*

- (i) Both $\tilde{P} := (L_1, L_2, \dots, L_g, L_1)$ and $\hat{P} := (\hat{L}_0, \hat{L}_1, \dots, \hat{L}_g)$ are $(g-2)$ -planes such that $(P, \tilde{P}) \in \bar{\Gamma}_{1,g-1}$ and $(P, \hat{P}) \in \bar{\Gamma}_{1,g-1}$.
- (ii) There exists a unique $(g-2)$ -plane $P' = (L'_0, L'_1, \dots, L'_{g-1}, L'_g)$ with $P' \notin \{\tilde{P}, \hat{P}\}$ and $(P, P') \in \bar{\Gamma}_{1,g-1}$.
- (iii) There exists a unique $(g-2)$ -plane P'' with $(P', P'') \in \bar{\Gamma}_{1,g-1}$ and $(P, P'') \in \bar{\Gamma}_{2,g-2}$.

Proof. Without loss of generality, we may assume that the first and second subscripts of y could be read modulo g . Note that (i) is obvious.

(ii) Let $y'_{0,0} \in \Gamma_{1,g-1}(y_{0,0}) \setminus \{y_{1,0}, y_{0,1}\}$. By Lemma 4.4 (ii), there exists a unique $(g-2)$ -line $L'_0 = (y'_{0,0}, y'_{0,1}, \dots, y'_{0,g})$ such that $(L_0, L'_0) \in \tilde{\Gamma}_{1,g-1}$. It follows from Lemma 4.3 (ii) that $\tilde{\partial}(y_{0,0}, y'_{0,1}) = (2, s)$. In view of $p_{(2,s),(g-1,1)}^{(1,g-1)} = 2$ and Lemma 4.4 (ii), there exist a unique vertex $y'_{1,0}$ and a unique $(g-2)$ -line $L'_1 = (y'_{1,0}, y'_{1,1}, \dots, y'_{1,g})$ such that $P_{(2,s),(g-1,1)}(y_{0,0}, y'_{0,0}) = \{y'_{1,0}, y'_{0,1}\}$ and $(L'_0, L'_1) \in \tilde{\Gamma}_{1,g-1}$. By Lemma 4.4 (iii), we can construct a unique $(g-2)$ -plane $P' = (L'_0, L'_1, \dots, L'_{g-1}, L'_g)$.

Since $(y_{0,0}, y'_{1,0}) \in \Gamma_{2,s}$, from Lemma 4.3 (i), we get $\partial(y_{1,0}, y'_{1,0}) = 1$. By Lemma 2.7 (iii) and Lemma 4.3 (ii), we obtain $\Gamma_{1,g-1}(y_{0,g-1}) = \{y_{0,0}, y'_{0,g-1}, y_{1,g-1}\}$. Since $\Gamma_{1,g-1}(y_{0,g-1}) = \Gamma_{2,s}(y'_{1,0})$, from the commutativity of Γ , we have $\tilde{\partial}(y_{0,g-1}, y'_{1,g-1}) = (2, s)$. Similarly, $(L_0, L'_1) \in \tilde{\Gamma}_{2,s}$. In view of Lemma 4.3 (i) again, we get $\partial(y_{1,i}, y'_{1,i}) = 1$, which implies $(L_1, L'_1) \in \tilde{\Gamma}_{1,g-1}$. By induction, (ii) is valid.

(iii) Let $P' = (L'_0, L'_1, \dots, L'_{g-1}, L'_g)$ with $L'_i = (y'_{i,0}, y'_{i,1}, \dots, y'_{i,g})$ for $0 \leq i \leq g$. By (ii), there exists a unique $(g-2)$ -plane P'' such that $(P', P'') \in \bar{\Gamma}_{1,g-1}$. Assume that $P'' = (L''_0, L''_1, \dots, L''_{g-1}, L''_g)$ with $L''_i = (y''_{i,0}, y''_{i,1}, \dots, y''_{i,g})$ and $y''_{0,0} \notin \{y'_{1,0}, y'_{0,1}\}$. Since $A_{1,g-1}^2 = A_{2,g-2} + 2A_{2,s}$, from the inductive hypothesis, one has $y''_{i,h} \notin \{y'_{i+1,h}, y'_{i,h+1}\}$ for $0 \leq i, h \leq g$. In view of Lemma 4.3 (ii), we get $y'_{i+1,h}, y'_{i,h+1} \in P_{(2,s),(g-1,1)}(y_{i,h}, y'_{i,h})$, which implies $\tilde{\partial}(y_{i,h}, y'_{i,h}) = (2, g-2)$ by $p_{(2,s),(g-1,1)}^{(1,g-1)} = 2$. This completes the proof of (iii). \square

Let P_0, P_1, \dots, P_j be $(g-2)$ -planes. We say that (P_0, P_1, \dots, P_j) is a *chain of $(g-2)$ -planes* if $(P_i, P_{i+1}) \in \bar{\Gamma}_{1,g-1}$ for $0 \leq i \leq j-1$, and $(P_{i'}, P_{i'+2}) \in \bar{\Gamma}_{2,g-2}$ for $0 \leq i' \leq j-2$. In particular, we say that a chain of $(g-2)$ -planes (P_0, P_1, \dots, P_j) is a *$(g-2)$ -cube* if $j = g$.

By Lemma 4.5 (ii) and (iii), there exists a $(g-2)$ -cube (P_0, P_1, \dots, P_g) with $P_0 = P_g$. In the remaining of this subsection, we set $P_i := (L_{i,0}, L_{i,1}, \dots, L_{i,g})$ and $L_{i,j} := (y(i, j, 0), y(i, j, 1), \dots, y(i, j, g))$ with $\Gamma_{1,g-1}(y(i, j, k)) = \{y(i+1, j, k), y(i, j+1, k), y(i, j, k+1)\}$ for $0 \leq i, j, k \leq g$, where the coordinates are taken modulo g .

Lemma 4.6 *The following hold:*

- (i) *For any fixed i , $y(i, j, k)$ s are pairwise distinct.*
- (ii) *For any fixed j , $y(i, j, k)$ s are pairwise distinct.*
- (iii) *For any fixed k , $y(i, j, k)$ s are pairwise distinct.*

Proof. We only need to prove (i). Suppose $y(i, 0, 0) = y(i, j, k)$ with $1 \leq j, k \leq g - 1$. Since $\partial(y(i, j, 0), y(i, 0, 0)) = \partial(y(i, j, 0), y(i, j, k)) = k$ and $(y(i, j, 0), y(i, j + 1, 0), \dots, y(i, 0, 0))$ is a path of length $g - j$, we obtain $g - j \geq k$. Observe that $C = (y(i, 0, 0), y(i, 1, 0), \dots, y(i, j, 0), y(i, j, 1), \dots, y(i, j, k - 1))$ is a circuit of length $j + k$. Then C is a minimal circuit and $\partial(y(i, j - 1, 0), y(i, j, 1)) = (2, g - 2)$, contrary to $\partial(y(i, j - 1, 0), y(i, j, 1)) = (2, s)$ by Lemma 4.3 (ii). \square

Proposition 4.7 *If $A_{1, g-1}^2 = A_{2, g-2} + 2A_{2, s}$, then Γ is isomorphic to the digraph in Theorem 1.1 (v).*

Proof. We prove it step by step.

Step 1 Show that $g < i + j + k < 2g$ if $y(a, b, c) = y(a + i, b + j, c + k)$ for some $a, b, c \in \{0, 1, \dots, g - 1\}$ and $i, j, k \in \{1, 2, \dots, g - 1\}$.

Note that $C = (y(a, b, c), y(a + 1, b, c), \dots, y(a + i, b, c), y(a + i, b + 1, c), \dots, y(a + i, b + j, c), y(a + i, b + j, c + 1), \dots, y(a + i, b + j, c + k - 1))$ and $C' = (y(a + i, b + j, c + k), y(a + i + 1, b + j, c + k), \dots, y(a, b + j, c + k), y(a, b + j + 1, c + k), \dots, y(a, b, c + k), y(a, b, c + k + 1), \dots, y(a, b, c - 1))$ are two circuits of length $i + j + k$ and $3g - i - j - k$, respectively. Hence, $g \leq i + j + k \leq 2g$. By Lemma 4.3 (ii), one gets $(y(a + i - 1, b, c), y(a + i, b + 1, c), (y(a - 1, b + j, c + k), y(a, b + j + 1, c + k))) \in \Gamma_{2, s}$, which implies that neither C nor C' is a minimal circuit. Thus, $g < i + j + k < 2g$.

Step 2 Show that $y(d, e, f) \in \{y(d + m, e + l, f + h), y(d + m, e + h, f + l)\}$ with $\{h, l, m\} = \{i, j, k\}$ for any $y(d, e, f)$ if $y(a, b, c) = y(a + i, b + j, c + k)$ for some $a, b, c \in \{0, 1, \dots, g - 1\}$ and $g \nmid i$.

By Lemma 4.6, we may assume $0 < i, j, k < g$. We only need to consider the case that $m = k$. Since $\partial(y(a + i, b + j, c), y(a, b, c)) = \partial(y(a + i, b + j, c), y(a + i, b + j, c + k)) = (k, g - k)$ and $y(a, b + j, c) \in P_{(g-i, i), (g-j, j)}(y(a + i, b + j, c), y(a, b, c))$, we get $p_{(g-i, i), (g-j, j)}^{(k, g-k)} \neq 0$, which implies $y(d, e, f) \in \Gamma_{i, g-i}(x)$ for some $x \in \Gamma_{j, g-j}(y(d + k, e, f)) = \{y(d + k + j, e, f), y(d + k, e + j, f), y(d + k, e, f + j)\}$. By Lemma 4.6, $y(d, e, f) \in \{y(d + k, e + j, f + i), y(d + k, e + i, f + j)\}$. The desired result holds.

Step 3 Show that $i = j = k = g/2$ if $y(0, 0, 0) = y(i, j, k)$ and $|\{i, j, k\}| \neq 3$ for some $i, j, k \in \{1, 2, \dots, g - 1\}$.

By Step 2, $y(i, j, k) = y(0, 0, 0) \in \{y(j, k, i), y(j, i, k)\} \cap \{y(k, j, i), y(k, i, j)\}$. Since $|\{i, j, k\}| \neq 3$, from Lemma 4.6, one gets $i = j = k$. Suppose $i \neq \frac{g}{2}$. In view of Step 2, one gets $y(0, 0, 0) = y(2i, 2i, 2i) = y(2i - g, 2i - g, 2i - g)$. By Step 1, we have $i < g/3$ or $2g/3 < i$, contrary to $g/3 < i < 2g/3$ from $y(0, 0, 0) = y(i, i, i)$. Hence, $i = g/2$.

Step 4 Show that $y(i, j, k) = y(2i, 2j, 2k) = y(i + k, j + i, k + j)$ if $y(0, 0, 0) = y(i, j, k)$ with $g \nmid i$ and $g \nmid (i - j)$.

By Step 3, we have $g \nmid (j - k)$ and $g \nmid (i - k)$. It follows from Step 2 that $y(i, j, k) \in \{y(2i, j + k, k + j), y(2i, 2j, 2k)\}$ and $y(i, j, k) \in \{y(i + k, 2j, i + k), y(i + k, j + i, k + j)\}$. If $y(i, j, k) = y(2i, j + k, k + j) = y(i + k, j + i, k + j)$ or $y(i, j, k) = y(2i, 2j, 2k) = y(i + k, 2j, i + k)$, then $g \mid (k - i)$, a contradiction. Suppose $y(0, 0, 0) = y(i, j, k) = y(2i, j + k, k + j) = y(i + k, 2j, i + k)$. By Step 3, we obtain $i + k \equiv 2j \equiv 0$ or $g/2 \pmod{g}$, and $j + k \equiv 2i \equiv 0$ or $g/2 \pmod{g}$. Since $g \nmid (i - j)$, one has $i + k \equiv 2j \equiv 0 \pmod{g}$ and $j + k \equiv 2i \equiv g/2 \pmod{g}$, or $i + k \equiv 2j \equiv g/2 \pmod{g}$ and $j + k \equiv 2i \equiv 0 \pmod{g}$. By $g \nmid i$ and Lemma 4.6, we have $g \nmid j$ and $g \nmid k$, which imply $j \equiv g/2 \pmod{g}$ or $i \equiv g/2 \pmod{g}$, contrary to $g \nmid (i - j)$. Thus, $y(i, j, k) = y(2i, 2j, 2k) = y(i + k, j + i, k + j)$.

Based on the above discussion, we complete the proof of this lemma as follows.

First, we will show that all vertices $y(d, e, f)$ with $0 \leq d, e, f \leq g - 1$ are distinct. Suppose $y(a, b, c) = y(a + i, b + j, c + k)$ for some $a, b, c \in \{0, 1, \dots, g - 1\}$ and $i, j, k \in \{1, 2, \dots, g - 1\}$. By Step 2, $y(0, 0, 0) \in \{y(i, j, k), y(i, k, j)\}$. Without loss of generality, we may assume $y(0, 0, 0) = y(i, j, k)$. In view of Step 2 again, we get $y(0, 0, 0) \in \{y(k, i, j), y(k, j, i)\}$ and $y(0, 0, 0) \in \{y(j, k, i), y(j, i, k)\}$. Suppose $y(0, 0, 0) \neq y(k, i, j)$ or $y(0, 0, 0) \neq y(j, k, i)$. If $y(i, j, k) = y(j, i, k) = y(k, i, j)$, from Lemma 4.6, then $i = k = j$ and $y(i, j, k) = y(j, k, i) = y(k, i, j)$. Similarly, $y(i, j, k) = y(j, k, i) = y(k, i, j)$ for the remaining two cases. Thus, $y(i, j, k) = y(j, k, i) = y(k, i, j)$.

Suppose $|\{i, j, k\}| = 3$. By Step 4, we have $y(0, 0, 0) = y(i + k, j + i, k + j) = y(2i + 2k, 2j + 2i, 2k + 2j)$. It follows from Step 2 that $y(i + k, j + i, k + j) \in \{y(i + j + k, i + j + k, i + j + k), y(i + j + k, 2i + j, 2k + j)\}$ and $y(i + j + k, 2i + j, 2k + j) \in \{y(i + j + 2k, 2i + 2j, 2k + j + i), y(i + j + 2k, 3i + j, 2k + 2j)\}$. Suppose $y(2i + 2k, 2j + 2i, 2k + 2j) = y(i + k, j + i, k + j) = y(i + j + k, 2i + j, 2k + j)$. By Lemma 4.6, we obtain $i = j$, contrary to $|\{i, j, k\}| = 3$. Hence, $y(0, 0, 0) = y(i + k, j + i, k + j) = y(i + j + k, i + j + k, i + j + k) = y(i + j + k - g, i + j + k - g, i + j + k - g)$. It follows from the Steps 1 and 3 that $i + j + k = 3g/2$ and $y(0, 0, 0) = y(i + k, j + i, k + j) = y(g/2 - j, g/2 - k, g/2 - i)$. Hence, $g/2 - j + g/2 - k + g/2 - i \equiv 0 \pmod{g}$, contrary to $g \nmid (g/2 - j + g/2 - k + g/2 - i)$ from Step 1. Hence, $|\{i, j, k\}| \neq 3$.

By Steps 2 and 3, we get $y(d, e, f) = y(d + g/2, e + g/2, f + g/2)$ for $0 \leq d, e, f \leq g - 1$. Then $|\{y(d, e, f) \mid 0 \leq d, e, f \leq g - 1\}| = g^3/2$. Observe $y(2, 1, 0), y(1, 1, 1) \in \Gamma_{3, 3g/2-3}(y(0, 0, 0))$. But we have $y(2, 0, 0) \in P_{(2, g-2), (1, g-1)}(y(0, 0, 0), y(2, 1, 0))$ and $P_{(2, g-2), (1, g-1)}(y(0, 0, 0), y(1, 1, 1)) = \emptyset$, a contradiction. Thus, all vertices $y(d, e, f)$ with $0 \leq d, e, f \leq g - 1$ are distinct.

Suppose $g > 3$. Observe $y(2, 2, 0), y(3, 1, 0) \in \Gamma_{4, 2g-4}(y(0, 0, 0))$. But $y(2, 0, 0) \in P_{(2, g-2), (2, g-2)}(y(0, 0, 0), y(2, 2, 0))$ and $P_{(2, g-2), (2, g-2)}(y(0, 0, 0), y(3, 1, 0)) = \emptyset$, a contradiction. The desired result holds. \square

Combining Propositions 2.2, 2.4, 2.6, 3.1, 3.8, 3.12, 4.2 and 4.7, we complete the proof of Theorem 1.1.

5 Proof of Lemma 4.1

In this section, we will prove Lemma 4.1 by contradiction. Suppose $\Gamma_{2, g-2} \notin \Gamma_{1, g-1}^2$. Since $k_{2, s} = 1$, from Lemma 2.7 (ii), we have $p_{(1, g-1), (1, g-1)}^{(2, s)} = 3$. In view of Lemma 2.1 (ii), $p_{(2, s), (g-1, 1)}^{(1, g-1)} = 1$.

Let $t = \min\{i \mid p_{(i+1, g-i-1), (g-1, 1)}^{(i, g-i)} = 1\}$. Note that $1 \leq t \leq g-1$. If $1 \leq j < t$, by $p_{(2, s), (g-1, 1)}^{(1, g-1)} = 1$, then $p_{(j+1, g-j-1), (g-1, 1)}^{(j, g-j)} = 2$; if $j \geq t$, then $p_{(j+1, g-j-1), (g-1, 1)}^{(j, g-j)} = 1$.

Pick a minimal circuit $(x_{0,0}, x_{1,0}, \dots, x_{g-1,0})$, where the first subscription of x could be read modulo g . Since $k_{2, s} = 1$, there exists a unique vertex $x_{i,j}$ such that $\tilde{\partial}(x_{i-1, j-1}, x_{i,j}) = (2, s)$ for $0 \leq i \leq g-1$ and $1 \leq j \leq g+2$. By $p_{(1, g-1), (1, g-1)}^{(2, s)} = 3$, $(x_{0,j}, x_{1,j}, \dots, x_{g-1,j})$ is a minimal circuit. Let $x'_{i,j}$ denote the vertex such that $\Gamma_{1, g-1}(x_{i,j}) = \Gamma_{g-1, 1}(x_{i+1, j+1}) = \{x_{i+1, j}, x_{i, j+1}, x'_{i,j}\}$ for $0 \leq i, j \leq g$.

By $p_{(2, s), (g-1, 1)}^{(1, g-1)} = 1$, we get $(x'_{i,j}, x'_{i+1, j+1}) \in \Gamma_{2, s}$ for $0 \leq i, j \leq g$. The fact that $p_{(t+1, g-t-1), (g-1, 1)}^{(t, g-t)} = 1$ implies $(x_{i,j}, x'_{i+t, j}), (x'_{i,j}, x_{i+t+1, j+1}) \notin \Gamma_{t+1, g-t-1}$. Since $p_{(1, g-1), (1, g-1)}^{(1, g-1)} = 0$, we obtain $\partial(x_{0,0}, x_{2,1}) = 3$. It follows from Lemma 2.1 (iii) that $\partial(x_{2,1}, x_{0,0}) = s-1$ and $\Gamma_{2, s}\Gamma_{1, g-1} = \{\Gamma_{3, s-1}\}$. Thus, $\partial(x_{0,0}, x_{2,2}) = 4$.

In the following, we give some useful results for distance of two vertices. Write $\tilde{f}_t := \tilde{\partial}(x_{0,1}, x'_{t,1})$ and $s_i := \partial(x_{1+i, 1+i}, x_{0,0})$ for $0 \leq i \leq 3$, where $s_0 = s$.

Lemma 5.1 *Let $(x_{3,3} = y_{0,j}, y_{1,j}, \dots, y_{s_j, j} = x_{2-j, 2-j})$ be a shortest path for $0 \leq j \leq 2$. If $\partial(x_{0,0}, x_{3+j, 1+j}) = 4 + 2j$ for some $j \in \{0, 1, 2\}$, then we have:*

- (i) $s_{j+1} \neq s_j - 2$ and $\tilde{\partial}(y_{i,j}, y_{i+2, j}) \neq (2, s)$ for $0 \leq i \leq s_j - 2$.
- (ii) $g \geq 2t$ and $|\Gamma_{t, g-t}\Gamma_{1, g-1}| = 3$.
- (iii) $\tilde{\partial}(y_{i-t, j}, y_{i+1, j}) = \tilde{f}_t$ for some $i \in \{t, t+1, \dots, s_j - 1\}$.

Proof. (i) Note that $\partial(x_{0,0}, x_{2+j, 1+j}) = 3 + 2j$ and $\tilde{\partial}(x_{0,0}, x_{1+j, 1+j}) = (2 + 2j, s_j)$. Since $k_{2, s} = 1$, from Lemma 2.1 (i), we have $k_{2+2j, s_j} = 1$. By Lemma 2.1 (iii), we get $x_{2+j, 1+j}, x_{1+j, 2+j}, x'_{1+j, 1+j} \in \Gamma_{3+2j, s_j-1}(x_{0,0})$. Hence, $\partial(x_{0,0}, x_{2+j, 2+j}) = 4 + 2j$.

Observe $\partial(x_{3+j, 1+j}, x_{0,0}) \geq s_j - 2$ and $s_{j+1} \geq s_j - 2$. Suppose $s_{j+1} = s_j - 2$. By Lemma 2.1 (i), one gets $k_{4+2j, s_{j+1}} = 1$, which implies $\partial(x_{3+j, 2+j}, x_{0,0}) = s_j - 3$ from Lemma 2.1 (iii). Then $s_j - 2 \leq \partial(x_{3+j, 1+j}, x_{0,0}) \leq 1 + \partial(x_{3+j, 2+j}, x_{0,0}) \leq s_j - 2$. But $x_{1+j, 1+j} \in P_{(2+2j, s_j), (2, s)}(x_{0,0}, x_{2+j, 2+j})$ and $P_{(2+2j, s_j), (2, s)}(x_{0,0}, x_{3+j, 1+j}) = \emptyset$, a contradiction. Therefore, $s_{j+1} \neq s_j - 2$.

Suppose $\tilde{\partial}(y_{i,j}, y_{i+2, j}) = (2, s)$ for some $i \in \{0, 1, \dots, s_j - 2\}$. By the commutativity of Γ , we may assume $i = 0$. Note that $y_{2,j} = x_{4,4}$ and $\partial(x_{4,4}, x_{2-j, 2-j}) = s_j - 2$.

Since $k_{2,s} = 1$, from Lemma 2.1 (iii), one has $\tilde{\partial}(x_{2-j,2-j}, x_{4,4}) = (4 + 2j, s_{1+j})$, contrary to $s_{1+j} \neq s_j - 2$. Thus, (i) is valid.

Let $\alpha_j = \max\{c \mid \tilde{\partial}(y_{0,j}, y_{c,j}) = (c, g - c)\}$. Note that $\alpha_j < s_j$ and there exists a minimal circuit $(y_{0,j} = v_g, y_{1,j}, \dots, y_{\alpha_j,j} = v_{\alpha_j}, v_{\alpha_j+1}, v_{\alpha_j+2}, \dots, v_{g-1})$.

(ii) and (iii) First, we prove $g \geq 2t$ and consider the case $t \geq 2$. Since $p_{(2,s),(g-1,1)}^{(1,g-1)} = 1$ and (i), we get $\alpha_j \geq t$. Suppose $\alpha_j \geq g - t + 1$. By $p_{(g-t,t),(1,g-1)}^{(g-t+1,t-1)} = p_{(t,g-t),(g-1,1)}^{(t-1,g-t+1)} = 2$, there exists a vertex $v_{\alpha_j-1} \in P_{(g-t,t),(1,g-1)}(y_{\alpha_j-g+t-1,j}, y_{\alpha_j,j}) \setminus \{y_{\alpha_j-1,j}\}$. Since $v_{\alpha_j-1}, y_{\alpha_j-1,j} \in \Gamma_{2,g-2}(y_{\alpha_j+1,j}) \cap \Gamma_{2,g-2}(v_{\alpha_j+1})$ and $p_{(2,s),(g-1,1)}^{(1,g-1)} = 1$, we get $y_{\alpha_j+1,j} = v_{\alpha_j+1}$, contrary to the fact that $(y_{0,j}, y_{1,j}, \dots, y_{\alpha_j,j} = v_{\alpha_j}, y_{\alpha_j+1,j} = v_{\alpha_j+1}, v_{\alpha_j+2}, \dots, v_{g-1})$ is a minimal circuit. Thus, $\alpha_j \leq g - t$ and $g \geq 2t$.

By (i), $\tilde{\partial}(y_{\alpha_j-t,j}, y_{\alpha_j,j}) = (t, g - t)$. The fact that $\partial(x_{0,1}, x_{t-1,1}) = t - 1$ and $\tilde{\partial}(x_{t-1,1}, x_{t,2}) = (2, s)$ imply $\tilde{\partial}(y_{\alpha_j-t,j}, y_{\alpha_j+1,j}) \neq \tilde{\partial}(x_{0,1}, x_{t,2})$. Since $y_{\alpha_j+1,j} \neq v_{\alpha_j+1}$ and $p_{(t+1,g-t-1),(g-1,1)}^{(t,g-t)} = 1$, one gets $\tilde{\partial}(y_{\alpha_j-t,j}, y_{\alpha_j+1,j}) \neq (t + 1, g - t - 1)$. Hence, $\tilde{\partial}(y_{\alpha_j-t,j}, y_{\alpha_j+1,j}) = f_t$. Thus, (ii) and (iii) hold. \square

Lemma 5.2 *The following hold:*

- (i) $g \geq 5$ and $\partial(x_{0,0}, x_{3,1}) = 4$.
- (ii) If $t \geq 2$, then $g \geq 6$, $\partial(x_{0,0}, x_{3,2}) = \partial(x_{0,0}, x_{4,1}) = 5$ and $\partial(x'_{0,0}, x'_{3,1}) + 2 = \partial(x_{0,0}, x_{4,2})$.

Proof. For $0 \leq i, j \leq g$, we claim that $(x'_{i,j}, x'_{i+t+1,j}) \in \Gamma_{t+1,g-t-1}$ if $g \geq 2t + 2$. Since $x_{i+t+1,j} \in P_{(t+1,g-t-1),(1,g-1)}(x_{i,j}, x'_{i+t+1,j})$, from the commutativity of Γ , we have $x'_{i,j}$ or $x_{i,j+1} \in P_{(1,g-1),(t+1,g-t-1)}(x_{i,j}, x'_{i+t+1,j})$. The fact $p_{(t+1,g-t-1),(1,g-1)}^{(t+2,g-t-2)} = p_{(g-t-2,t+2),(g-1,1)}^{(g-t-1,t+1)} = 1$ implies that $x'_{i+t+1,j} \notin P_{(t+1,g-t-1),(1,g-1)}(x_{i,j+1}, x_{i+t+2,j+1})$. Thus, our claim is valid.

(i) Suppose $g = 3$. Since $p_{(2,1),(2,1)}^{(1,2)} = p_{(1,2),(1,2)}^{(2,1)} = 1$, $t = 1$. By Lemma 2.7 (ii) and Lemma 2.1 (iv), we have $A_{1,2}^2 = A_{2,1} + A_{2,l} + 3A_{2,s}$ with $1 < l < s \leq 4$. From $p_{(1,2),(1,2)}^{(2,s)} = 3$, we get $\tilde{\partial}(x_{0,0}, x_{2,1}) = (3, 3)$ and $s = 4$. Since $\Gamma_{2,4}\Gamma_{1,2} = \{\Gamma_{3,3}\}$, $\partial(x_{0,2}, x_{0,0}) = 1$. By $(x_{0,0}, x_{1,1}) \in \Gamma_{2,4}$, $(x_{0,0}, x'_{0,1}) \in \Gamma_{2,l}$. The fact that $(x'_{2,0}, x'_{0,1}) \in \Gamma_{2,4}$ implies $(x_{2,1}, x'_{0,1}) \in \Gamma_{2,1}$, contrary to $(x_{2,1}, x'_{0,1}) \in \Gamma_{2,l}$. Hence, $g \neq 3$.

Suppose $\partial(x_{0,0}, x_{3,1}) < 4$. Since $\partial(x_{0,0}, x_{2,1}) = \partial(x_{0,0}, x_{3,0}) = 3$, $\partial(x_{0,0}, x'_{2,0}) < 3$. By $p_{(1,g-1),(g-1,1)}^{(2,g-2)} = 0$, one has $\partial(x_{0,0}, x'_{2,0}) = 2$. The fact that $p_{(1,g-1),(g-1,1)}^{(1,g-1)} = 0$ and $\partial(x_{0,1}, x'_{2,0}) \geq 2$ imply $\partial(x'_{0,0}, x'_{2,0}) = 1$. Similarly, $\partial(x'_{2,0}, x'_{4,0}) = 1$. Then $g > 4$. Since $P_{(3,g-3),(g-1,1)}(x_{0,0}, x_{2,0}) = \{x_{3,0}\}$, from the claim, we get $t = 2$.

Since $p_{(1,g-1),(1,g-1)}^{(2,s)} = 3$, we have $x'_{2,0} \in P_{(1,g-1),(1,g-1)}(x'_{0,0}, x'_{1,1})$. The fact $\partial(x'_{0,0}, x_{4,0}) \neq 1$ implies $(x'_{0,0}, x'_{4,0}) \notin \Gamma_{2,s}$ and $x'_{1,1} \neq x'_{4,0}$. By $p_{(1,g-1),(1,g-1)}^{(1,g-1)} = 0$, one gets $\Gamma_{1,g-1}(x'_{2,0}) = \{x'_{1,1}, x'_{4,0}, x_{3,1}\}$ and $x'_{2,0} \neq x_{1,1}$. Hence, $(x_{0,0}, x'_{2,0}) \in \Gamma_{2,g-2}$ and $\Gamma_{2,g-2} \in \Gamma_{2,g-2}\Gamma_{1,g-1}$. Observe $x_{1,1} \in P_{(2,s),(1,g-1)}(x_{0,0}, x'_{1,1})$ and $x_{1,1} \in P_{(2,s),(2,g-2)}(x_{0,0}, x_{3,1})$. By $p_{(1,g-1),(g-1,1)}^{(1,g-1)} = p_{(1,g-1),(g-1,1)}^{(2,g-2)} = 0$, we obtain $x_{3,1}, x'_{1,1} \notin \Gamma_{2,g-2}(x_{0,0})$, which implies $\tilde{\partial}(x_{0,0}, x'_{4,0}) = (2, g - 2)$. Since $\partial(x_{0,1}, x'_{4,0}) \geq 4$, one has

$\partial(x_{1,0}, x'_{4,0}) = 1$. From $p_{(1,g-1),(1,g-1)}^{(2,s)} = 3$, we get $x'_{4,0} \in P_{(1,g-1),(1,g-1)}(x_{1,0}, x_{2,1})$, which implies $x_{2,1} \in P_{(1,g-1),(g-1,1)}(x_{2,0}, x'_{4,0})$, contrary to $p_{(1,g-1),(g-1,1)}^{(2,g-2)} = 0$ or $p_{(1,g-1),(1,g-1)}^{(2,s)} = 3$. Then $\partial(x_{0,0}, x_{3,1}) = 4$.

Suppose $g = 4$. Note that $3 \leq s \leq 6$ and $\tilde{\partial}(x_{0,0}, x_{2,1}) = (3, s-1)$. Since $p_{(1,3),(1,3)}^{(2,s)} = 3$ and $p_{(1,3),(1,3)}^{(1,3)} = 0$, we have $s \neq 3$. By Lemma 5.1 (i), $s_1 > s-2$ and $(x_{3,2}, x_{0,0})$ is not an arc, which imply $s \neq 4$ from $\Gamma_{2,s}\Gamma_{1,3} = \{\Gamma_{3,s-1}\}$. Since $x_{1,1} \in P_{(2,s),(2,2)}(x_{0,0}, x_{3,1})$ and $P_{(2,s),(2,2)}(x_{0,0}, x_{2,2}) = \emptyset$, one gets $\partial(x_{2,2}, x_{0,0}) \neq \partial(x_{3,1}, x_{0,0})$. If $\partial(x_{3,1}, x_{0,0}) = 4$, then $x_{0,2} \in P_{(2,s),(2,2)}(x_{3,1}, x_{0,0})$, which implies $\partial(x_{2,2}, x_{0,0}) < 4$, contrary to $s_1 > s-2$; if $\partial(x_{3,1}, x_{0,0}) = 3$, then $s = 5$, which implies $x_{0,2} \in P_{(2,5),(1,3)}(x_{3,1}, x_{0,0})$, contrary to $g = 4$. Thus, (i) holds.

(ii) Observe $x_{1,1} \in \Gamma_{2,s}(x_{0,0}) \cap \Gamma_{g-2,2}(x_{3,1}) \cap \Gamma_{g-2,2}(x'_{2,1})$. It follows from (i) and Lemma 2.1 (iii) that $\partial(x_{0,0}, x_{3,2}) = 5$.

Suppose $g = 5$. (i) and Lemma 5.1 (ii) imply $t = 2$. By Lemma 5.1 (i), $s-2 < \partial(x_{2,2}, x_{0,0}) \leq 6$. Hence, $4 \leq s \leq 7$ and $\Gamma_{4,s-2} \notin \Gamma_{2,s}^2$. In view of Lemma 2.1 (iii), one gets $\Gamma_{2,s}\Gamma_{2,3} = \{\Gamma_{4,s-2}\}$ and $(x_{0,0}, x_{3,1}) \in \Gamma_{4,s-2}$. Since $p_{(1,4),(1,4)}^{(2,s)} = 3$ and $p_{(1,4),(1,4)}^{(1,4)} = 0$, we obtain $s \neq 4$. By $\Gamma_{2,s}\Gamma_{1,g-1} = \{\Gamma_{3,s-1}\}$ and $p_{(1,4),(4,1)}^{(2,3)} = 0$, one has $s \neq 5$. Since $\partial(x_{2,2}, x_{0,0}) > s-2$, we get $\partial(x_{4,2}, x_{0,0}) > s-4$, which implies $s = 7$ from $\Gamma_{2,s}\Gamma_{2,3} = \{\Gamma_{4,s-2}\}$.

By Lemma 5.1 (i), we get $\tilde{\partial}(x_{0,0}, x_{2,2}) = (4, 6)$. It follows from Lemma 2.1 (i) and (iii) imply $k_{4,6} = 1$ and $\tilde{\partial}(x_{0,0}, x_{3,2}) = (5, 5)$. Since $x_{2,2} \in P_{(4,6),(1,4)}(x_{0,0}, x_{3,2})$, we obtain $\partial(x_{0,4}, x_{0,0}) = 1$. By Lemma 5.1 (ii), $\Gamma_{1,4}\Gamma_{2,3} = \{\Gamma_{3,2}, \Gamma_{3,6}, \Gamma_{\tilde{f}_2}\}$. Since $x_{0,4} \in P_{(2,7),(1,4)}(x_{4,3}, x_{0,0})$ and $\Gamma_{2,7}\Gamma_{1,4} = \{\Gamma_{3,6}\}$, $(x'_{4,2}, x_{0,0}) \in \Gamma_{\tilde{f}_2}$. The fact that $x_{1,1} \in \Gamma_{2,7}(x_{0,0}) \cap \Gamma_{2,3}(x_{4,1}) \cap \Gamma_{\tilde{f}_2}(x'_{3,1})$, $x_{0,2} \in \Gamma_{2,7}(x_{4,1}) \cap \Gamma_{2,3}(x_{0,0})$ and $x'_{4,2} \in \Gamma_{2,7}(x'_{3,1}) \cap \Gamma_{\tilde{f}_2}(x_{0,0})$ imply $\partial(x_{0,0}, x_{4,1}) = \partial(x_{4,1}, x_{0,0})$ and $\partial(x_{0,0}, x'_{3,1}) = \partial(x'_{3,1}, x_{0,0})$. Hence, $\partial(x'_{3,1}, x_{0,0}) = 3$ or $\partial(x_{4,1}, x_{0,0}) = 3$, contrary to $s = 7$. Then $g \geq 6$.

If $t = 2$, from the claim and Lemma 5.1 (iii), then $\partial(x'_{0,0}, x'_{3,0}) = \partial(x_{1,0}, x'_{3,0}) = 3$, which implies $\partial(x_{0,0}, x'_{3,0}) = 4$ since $\partial(x_{0,1}, x'_{3,0}) \geq 3$; if $t \geq 3$, then $\partial(x_{0,0}, x'_{3,0}) = 4$. By $\tilde{\partial}(x_{0,0}, x_{4,0}) = (4, g-4)$, one has $\partial(x_{0,0}, x_{4,1}) = 5$. Hence, $\partial(x_{0,0}, x_{4,2}) = \partial(x_{0,0}, x'_{3,1}) + 1$. Since $x_{2,1} \in P_{(2,s),(2,g-2)}(x_{1,0}, x'_{3,1})$, from (i) and Lemma 2.1 (iii), we get $\partial(x_{1,0}, x'_{3,1}) = 4$. The fact that $\tilde{\partial}(x_{0,0}, x'_{3,0}) = \tilde{\partial}(x_{0,1}, x'_{3,1})$ implies $\partial(x_{0,0}, x_{4,2}) = \partial(x'_{0,0}, x'_{3,1}) + 2$. Thus, (ii) is valid. \square

By Lemma 2.1 (i), one has $k_{2+2j,s_j} = 1$ for $0 \leq j \leq r$, where $r = \max\{c \mid \partial(x_{0,0}, x_{3+c,1+c}) = 4+2c \text{ and } 0 \leq c \leq 2\}$. Lemma 2.1 (iii) implies $|\Gamma_{2+2j,s_j}\Gamma_{t,g-t}| = 1$. In view of Lemma 5.1 (i), we can pick a shortest path $(x_{3,3} = y_{0,j}, y_{1,j}, \dots, y_{s_j,j} = x_{2-j,2-j})$ such that $\tilde{\partial}(y_{0,0}, y_{t,0}) = (t, g-t)$ and $y_{i,0} = y_{i,1} = \dots = y_{i,r}$ for $0 \leq i \leq t$.

Lemma 5.3 Suppose $\partial(x_{0,0}, x_{3+j,1+j}) = 4 + 2j$ for $j = 1$ or 2 . If there exist two vertices u_1 and $y_{i,j'}$ such that $\partial(u_1, y_{i,j'}) = \partial(y_{1,j'}, y_{i,j'})$ with $u_1 \neq y_{1,j'}$, then $\tilde{\partial}(x_{3,3}, y_{i,j'}) \neq \tilde{\partial}(x_{3,3}, y_{i,j''})$ for $0 \leq j', j'' \leq j$ and $j' \neq j''$.

Proof. Suppose for the contrary that $\tilde{\partial}(x_{3,3}, y_{i,j'}) = \tilde{\partial}(x_{3,3}, y_{i,j''})$. Since $k_{2+2j'',s_{j''}} = k_{2+2j',s_{j'}} = 1$, from Lemma 2.1 (iii), there exists a shortest path $(y_{0,j'}, y_{1,j'}, \dots, y_{i,j'} =$

$v_i, v_{i+1}, \dots, v_{s_{j''}} = x_{2-j'', 2-j''}$.

By the commutativity of Γ , there exist distinct vertices u_{i-1} and v_{i-1} such that $u_{i-1}, v_{i-1} \in \Gamma_{g-1,1}(y_{i,j'})$ and $\partial(x_{3,3}, u_{i-1}) = \partial(x_{3,3}, v_{i-1}) = i-1$. It follows from Lemma 5.1 (i) that $y_{i+1,j'}, v_{i+1} \notin \Gamma_{2,s}(u_{i-1}) \cup \Gamma_{2,s}(v_{i-1})$. Since $p_{(2,s),(g-1,1)}^{(1,g-1)} = 1$, $y_{i+1,j'} = v_{i+1}$. By induction, we have $y_{c,j'} = v_c$ for $i \leq c \leq \min\{s_{j'}, s_{j''}\}$. Without loss of generality, we may assume $s_{j'} < s_{j''}$. Since $k_{2+2j',s_{j'}} = 1$, from Lemma 2.1 (iii), we get $|\Gamma_{s_{j'}, 2+2j'} \Gamma_{g-1,1}| = 1$ and $\partial(x_{3,3}, v_{s_{j'}-1}) = \partial(x_{3,3}, b_{s_{j'}-1})$ with $b_{s_{j'}-1} \in P_{(g-1,1),(2,s)}(v_{s_{j'}}, v_{s_{j'}+1})$, contrary to Lemma 5.1 (i). The desired result holds. \square

Write $\tilde{d}_t := \tilde{\partial}(x_{0,1}, x'_{t+1,1})$ and $\tilde{h}_t := \tilde{\partial}(x'_{0,0}, x'_{t+1,1})$.

Lemma 5.4 *If $\partial(x_{0,0}, x_{4,2}) = 6$, then $\partial(x_{0,0}, x_{5,3}) < 8$, $\tilde{d}_t \neq \tilde{h}_t$, $\tilde{\partial}(y_{0,j}, y_{t+2,j}) = \tilde{h}_t$ and $\tilde{\partial}(y_{i,j'}, y_{i+t+2,j'}) = \tilde{d}_t$ for some $i \in \{0, 1, \dots, s_{j'} - t - 2\}$ with $\{j, j'\} = \{0, 1\}$.*

Proof. Let $r = \max\{c \mid \partial(x_{0,0}, x_{3+c,1+c}) = 4 + 2c \text{ and } 1 \leq c \leq 2\}$.

For $0 \leq \mu \leq r$, we claim $(y_{\lambda,\mu}, y_{\lambda+t+2,\mu}) \in \Gamma_{\tilde{d}_t}$ for some $\lambda \in \{0, 1, \dots, s_\mu - t - 2\}$ if $(y_{\lambda',\mu}, y_{\lambda'+t+1,\mu}) \notin \Gamma_{\tilde{f}_t}$ for some $\lambda' \in \{0, 1, \dots, s_\mu - t - 1\}$. It follows from Lemma 5.1 (i) that $\tilde{\partial}(y_{\lambda'',\mu}, y_{\lambda''+t,\mu}) = (t, g-t)$ for $0 \leq \lambda'' \leq s_\mu - t$. In view of $k_{2+\mu,s_\mu} = 1$, Lemma 2.1 (iii) and the commutativity of Γ , we may assume $\lambda' = 0$. Since $\tilde{\partial}(x_{0,0}, x_{1,1}) = (2, s)$ and $\partial(x_{1,1}, x_{t,1}) = t-1$, we have $(y_{0,\mu}, y_{t+1,\mu}) \in \Gamma_{t+1,g-t-1}$. By Lemma 5.1 (iii), there exists an integer λ such that $\lambda+1 = \min\{c \mid \tilde{\partial}(y_{c,\mu}, y_{c+t+1,\mu}) = \tilde{f}_t\}$. Observe $0 \leq \lambda \leq s_\mu - t - 2$ and $(y_{\lambda,\mu}, y_{\lambda+t+1,\mu}) \in \Gamma_{t+1,g-t-1}$. Since $x_{1,2} \in P_{(2,s),(t,g-t)}(x_{0,1}, x_{t+1,2})$, from Lemma 5.1 (i), we get $\tilde{\partial}(y_{\lambda,\mu}, y_{\lambda+t+2,\mu}) = \tilde{d}_t$. Thus, the claim is valid.

Let $0 \leq \mu, \mu' \leq r$ and $\mu \neq \mu'$. Suppose $(y_{\lambda,\mu}, y_{\lambda+t+1,\mu}), (y_{\lambda',\mu'}, y_{\lambda'+t+1,\mu'}) \in \Gamma_{\tilde{f}_t}$ for $0 \leq \lambda \leq s_\mu - t - 1$ and $0 \leq \lambda' \leq s_{\mu'} - t - 1$. Lemma 5.1 (i) implies that $(y_{\lambda,\mu}, y_{\lambda+t,\mu}), (y_{\lambda',\mu'}, y_{\lambda'+t,\mu'}) \in \Gamma_{t,g-t}$ for $0 \leq \lambda \leq s_\mu - t$ and $0 \leq \lambda' \leq s_{\mu'} - t$. Since $p_{\tilde{f}_t,(g-1,1)}^{(t,g-t)} = 1$ from Lemma 5.1 (ii), we get $y_{a,\mu} = y_{a,\mu'}$ for $0 \leq a \leq \min\{s_\mu, s_{\mu'}\}$. Without loss of generality, we may assume $s_\mu < s_{\mu'}$. Since $k_{2+2\mu,s_\mu} = 1$, from Lemma 2.1 (iii), we obtain $|\Gamma_{s_\mu, 2+2\mu} \Gamma_{g-1,1}| = 1$ and $\partial(x_{3,3}, y_{s_\mu-1,\mu'}) = \partial(x_{3,3}, b_{s_\mu-1})$ with $b_{s_\mu-1} \in P_{(g-1,1),(2,s)}(y_{s_\mu,\mu'}, y_{s_\mu+1,\mu'})$, contrary to Lemma 5.1 (i).

Suppose that $(y_{\lambda,\mu}, y_{\lambda+t+1,\mu}) \notin \Gamma_{\tilde{f}_t}$ and $(y_{\lambda',\mu'}, y_{\lambda'+t+1,\mu'}) \notin \Gamma_{\tilde{f}_t}$ for some $\lambda \in \{0, 1, \dots, s_\mu - t - 1\}$ and $\lambda' \in \{0, 1, \dots, s_{\mu'} - t - 1\}$. In view of the claim, Lemma 2.1 (iii) and the commutativity of Γ , we may assume $y_{t+2,\mu}, y_{t+2,\mu'} \in \Gamma_{\tilde{d}_t}(x_{3,3})$. Observe $x_{t+1,1} \in P_{(t+1,g-t-1),(1,g-1)}(x_{0,1}, x'_{t+1,1})$ and $x_{1,1} \in P_{(1,g-1),\tilde{f}_t}(x_{0,1}, x'_{t+1,1})$, contrary to Lemma 5.3. By the claim again, we get $r = 1$ and $\partial(x_{0,0}, x_{5,3}) < 8$. Then $(y_{i,j}, y_{i+t+1,j}) \in \Gamma_{\tilde{f}_t}$ for $0 \leq i \leq s_j - t - 1$, and $(y_{i',j'}, y_{i'+t+2,j'}) \in \Gamma_{\tilde{d}_t}$ for some $i' \in \{0, 1, \dots, s_{j'} - t - 2\}$ with $\{j, j'\} = \{0, 1\}$.

By Lemma 5.1 (iii) and Lemma 5.3, one has $p_{(1,g-1),(t,g-t)}^{\tilde{f}_t} = 1$ and $|\Gamma_{\tilde{f}_t} \Gamma_{g-1,1}| \neq 1$. In view of Lemma 2.1 (iii), we obtain $k_{\tilde{f}_t} \neq 1$, which implies $t+1 > s_j$ by $k_{2+2j,s_j} = 1$. Since $(x_{1,1}, x'_{t+1,1}) \in \Gamma_{\tilde{f}_t}$ and $x_{2,1} \in P_{(2,s),(t,g-t)}(x_{1,0}, x'_{t+1,1})$, from Lemma 5.1 (i), we get $\tilde{\partial}(y_{0,j}, y_{t+2,j}) \neq \tilde{\partial}(x_{1,0}, x'_{t+1,1})$. Observe $x_{1,1} \in P_{(1,g-1),\tilde{f}_t}(x_{0,1}, x'_{t+1,1})$ and $x_{t+1,1} \in P_{(t+1,g-t-1),(1,g-1)}(x_{0,1}, x'_{t+1,1})$. By Lemma 5.3, Lemma 2.1 (iii) and the commutativity of Γ , we have $\tilde{\partial}(y_{0,j}, y_{t+2,j}) = \tilde{h}_t$ and $\tilde{h}_t \neq \tilde{d}_t$. \square

Proof of Lemma 4.1.

Case 1. $t = 1$.

By Lemma 2.7 (ii), $A_{1,g-1}^2 = A_{2,g-2} + A_{2,l} + 3A_{2,s}$ with $g - 2 < l < s$. Lemma 5.2 (i) implies $g \geq 5$. It follows from the claim in Lemma 5.2 that $\tilde{\partial}(x'_{0,2}, x'_{2,2}) = \tilde{\partial}(x'_{1,1}, x'_{3,1}) = (2, g - 2)$ and there exist two paths $(x'_{0,2}, z_0, x'_{2,2})$ and $(x'_{1,1}, z_1, x'_{3,1})$. Since $p_{(1,g-1),(1,g-1)}^{(2,s)} = 3$, we get $z_0 \in P_{(1,g-1),(1,g-1)}(x'_{1,1}, x'_{2,2})$. By $p_{(g-1,1),(2,g-2)}^{(1,g-1)} = 1$, we have $\tilde{\partial}(x'_{3,1}, x'_{5,2}) \neq (2, g - 2)$, which implies $\partial(x'_{0,2}, x'_{3,1}) > 2$ and $z_0 \neq z_1$. Since $p_{(1,g-1),(1,g-1)}^{(2,g-2)} = 1$, we obtain $\Gamma_{1,g-1}(x'_{1,1}) = \{z_0, z_1, x_{2,2}\}$.

Since $x_{4,1} \in P_{(3,g-3),(1,g-1)}(x_{1,1}, x'_{4,1})$, from the commutativity of Γ , we have $x_{1,2}$ or $x'_{1,1} \in P_{(1,g-1),(3,g-3)}(x_{1,1}, x'_{4,1})$. The fact that $p_{(g-1,1),(2,g-2)}^{(1,g-1)} = 1$ implies $(x'_{4,1}, x_{6,2}) \notin \Gamma_{2,g-2}$ and $x_{1,2} \notin \Gamma_{g-3,3}(x'_{4,1})$. Then $x'_{1,1} \in \Gamma_{g-3,3}(x'_{4,1})$.

Pick a path $(x'_{1,1}, w_0, w_1, x'_{4,1})$. Since $(x'_{4,1}, x_{6,2}) \notin \Gamma_{2,g-2}$, $w_0 \in \{z_0, z_1\}$. Observe $(x'_{1,1}, x'_{3,1}) \in \Gamma_{2,g-2}$ and $(x_{1,1}, x'_{3,1}) \notin \Gamma_{3,g-3}$. By $p_{(2,g-2),(g-1,1)}^{(1,g-1)} = 1$, we have $(x_{1,1}, z_1) \in \Gamma_{2,l}$, which implies $(x_{1,1}, z_0) \in \Gamma_{2,g-2}$. Since $(x_{1,1}, x'_{4,1}) \notin \Gamma_{4,g-4}$, one gets $w_0 = z_1$. By $p_{(2,g-2),(g-1,1)}^{(1,g-1)} = 1$ again, we obtain $w_1 = x'_{3,1}$, which implies $x_{4,1}, x'_{3,1} \in P_{(1,g-1),(1,g-1)}(x_{3,1}, x'_{4,1})$, contrary to $p_{(1,g-1),(1,g-1)}^{(2,l)} = 1$.

Case 2. $t = 2$.

We claim $|\Gamma_{\tilde{f}_2} \Gamma_{1,g-1}| = 3$ and $\Gamma_{\tilde{h}_2} \notin \Gamma_{3,g-3} \Gamma_{1,g-1}$. By Lemma 5.1 (i), Lemma 5.2 (i) and Lemma 2.1 (iii), we get $\Gamma_{2,s} \Gamma_{2,g-2} = \{\Gamma_{4,s-2}\}$. Then $(x_{0,1}, x_{3,2}), (x_{1,0}, x'_{3,1}) \in \Gamma_{4,s-2}$. Lemma 5.2 (ii) implies $g \geq 6$ and $p_{(2,g-2),(1,g-1)}^{(3,g-3)} = p_{(g-2,2),(g-1,1)}^{(g-3,3)} = 1$. Hence, $\tilde{d}_2 \neq (4, s - 2)$ and $\Gamma_{3,g-3} \Gamma_{1,g-1} = \{\Gamma_{4,g-4}, \Gamma_{4,s-2}, \Gamma_{\tilde{d}_2}\}$. Since $(x_{1,1}, x'_{3,1}) \in \Gamma_{\tilde{f}_2}$, we obtain $\Gamma_{4,s-2}, \Gamma_{\tilde{d}_2} \in \Gamma_{\tilde{f}_2} \Gamma_{1,g-1}$. If $\partial(x_{0,0}, x_{4,2}) = 6$, by Lemma 5.1 (i) and Lemma 5.4, then $\tilde{h}_2 \notin \{(4, g - 4), (4, s - 2), \tilde{d}_2\}$; if $\partial(x_{0,0}, x_{4,2}) < 6$, by Lemma 5.2 (ii), then $\partial(x_{0,1}, x'_{3,1}) = \partial(x_{0,0}, x'_{3,0}) = 4$ and $\partial(x'_{0,0}, x'_{3,1}) < 4$. The claim is valid.

In view of Lemma 2.7 (ii), we have $A_{1,g-1}^2 = A_{2,g-2} + 3A_{2,s}$. By Lemma 2.1 (i), $k_{2,g-2} = 6$. Since $p_{(2,g-2),(1,g-1)}^{(3,g-3)} = 1$, from Lemma 5.1 (ii) and Lemma 2.1 (vi), we get $k_{3,g-3} = 6$. The fact that $\Gamma_{2,s} \Gamma_{1,g-1} = \{\Gamma_{3,s-1}\}$ implies $k_{3,s-1} = 3$.

By the claim in Lemma 5.2, one has $\tilde{\partial}(x'_{0,1}, x'_{3,1}) = (3, g - 3)$ and there exists a path $(x'_{0,1}, u_1, u_2, x'_{3,1})$. From $p_{(1,g-1),(1,g-1)}^{(2,s)} = 3$, $u_2 \in P_{(1,g-1),(1,g-1)}(x'_{2,0}, x'_{3,1})$. Since $p_{(1,g-1),(2,g-2)}^{(3,g-3)} = 1$, one gets $u_1 \neq x_{1,2}$ and $\tilde{\partial}(x_{0,1}, u_1) = (2, g - 2)$. By $(x_{0,1}, x'_{3,1}) \notin \Gamma_{4,g-4}$, $p_{(3,g-3),(g-1,1)}^{(2,g-2)} = 1$ and $\Gamma_{2,s} \Gamma_{1,g-1} = \{\Gamma_{3,s-1}\}$, we obtain $\tilde{\partial}(x_{0,1}, u_2) = \tilde{\partial}(x_{1,1}, x'_{3,1}) = \tilde{f}_2$. Since $\Gamma_{2,s} \Gamma_{2,g-2} = \{\Gamma_{4,s-2}\}$, we have $(x'_{g-1,0}, u_2) \in \Gamma_{4,s-2}$, which implies $\tilde{\partial}(x_{-1,1}, u_2) = \tilde{h}_2$ or \tilde{d}_2 from $(x_{1,0}, x'_{3,1}) \in \Gamma_{4,s-2}$.

Case 2.1. $\partial(x_{0,0}, x_{4,2}) < 6$.

From Lemma 5.2 (ii), $\partial(x'_{0,0}, x'_{3,1}) < 4$. Since $\Gamma_{2,s} \Gamma_{2,g-2} = \{\Gamma_{4,s-2}\}$, we get $(x'_{1,1}, x'_{3,1}) \notin \Gamma_{2,g-2}$. By $p_{(2,g-2),(1,g-1)}^{(3,g-2)} = 1$, one has $(x_{1,2}, x'_{3,1}) \notin \Gamma_{2,g-2}$. Hence, $p_{(1,g-1),(2,g-2)}^{\tilde{f}_2} = 1$. Lemma 5.1 (ii) and Lemma 2.1 (vi) imply $k_{\tilde{f}_2} = 6$.

Since $k_{2,s} = 1$, from Lemma 2.1 (iii) and Lemma 5.1 (iii), we may assume $\tilde{\partial}(y_{0,0}, y_{3,0}) = \tilde{\partial}(x_{1,1}, x'_{3,1})$. By Lemma 5.1 (i) and $\partial(x'_{0,0}, x'_{3,1}) < 4$, $\tilde{\partial}(y_{0,0}, y_{4,0}) = \tilde{d}_2$.

Observe $x_{1,1} \in P_{(2,s),\tilde{f}_2}(x_{0,0}, x'_{3,1})$ and $x'_{0,1} \in P_{(2,s),(3,g-3)}(x'_{g-1,0}, x'_{3,1})$. Since $s > 4$, we get $\tilde{\partial}(y_{0,0}, y_{5,0}) = \tilde{\partial}(x_{-1,1}, x'_{3,1})$, which implies $\tilde{\partial}(x_{-1,1}, u_2) = \tilde{d}_2$ by $\partial(x'_{0,0}, x'_{3,1}) < 4$. Hence, $\tilde{\partial}(x_{0,0}, u_2) = \tilde{h}_2$.

Since $\Gamma_{2,s}\Gamma_{1,g-1} = \{\Gamma_{3,s-1}\}$ and $p_{(1,g-1),(1,g-1)}^{(2,g-2)} = 1$, from the claim, we have $\tilde{\partial}(x_{0,0}, u_1) = \tilde{f}_2$, which implies $u_1, x'_{2,0} \in P_{\tilde{f}_2,(1,g-1)}(x_{0,0}, u_2)$. By $\partial(x'_{0,0}, x'_{3,1}) < 4$ and Lemma 2.1 (vi), we obtain $p_{\tilde{f}_2,(1,g-1)}^{\tilde{h}_2} = 2$ and $k_{\tilde{h}_2} = 3$. Hence, $\tilde{h}_2 = (1, g-1)$ or $(3, s-1)$. If $\tilde{h}_2 = (1, g-1)$, by Lemma 2.1 (i), then $\tilde{f}_2 = (3, 3)$ since $k_{\tilde{f}_2} = 6$ and $x'_{0,0} \in P_{(g-1,1),(1,g-1)}(x_{1,1}, x'_{3,1})$, which implies $g < 6$, a contradiction. If $\tilde{h}_2 = (3, s-1)$, then $\partial(x'_{1,1}, x'_{3,1}) = 1$, contrary to $\partial(y_{0,0}, y_{3,0}) = \partial(x_{1,1}, x'_{3,1}) = 3$.

Case 2.2. $\partial(x_{0,0}, x_{4,2}) = 6$.

In view of Lemma 5.1 (iii) and Lemma 5.3, we obtain $p_{(1,g-1),(2,g-2)}^{\tilde{f}_2} = 1$. It follows from Lemma 5.1 (ii) and Lemma 2.1 (vi) that $k_{\tilde{f}_2} = 6$. Since $g \geq 6$, we have $p_{(3,g-3),(1,g-1)}^{(4,g-4)} = p_{(g-3,3),(g-1,1)}^{(g-4,4)} = 1$, which implies $\partial(x_{0,2}, x'_{3,1}) > 3$ and $p_{(1,g-1),\tilde{f}_2}^{\tilde{d}_2} = 1$. In view of the claim and Lemma 2.1 (vi) again, one gets $k_{\tilde{d}_2} = 6$.

Note that $k_{2,s} = k_{4,s_1} = 1$. By Lemma 5.4, Lemma 2.1 (iii) and the commutativity of Γ , we may assume $(y_{0,j}, y_{4,j}) \in \Gamma_{\tilde{h}_2}$ and $(y_{0,j'}, y_{4,j'}) \in \Gamma_{\tilde{d}_2}$ with $\{j, j'\} = \{0, 1\}$. Observe $x_{1,1} \in P_{(2,s),\tilde{f}_2}(x_{0,0}, x'_{3,1})$ and $x'_{0,1} \in P_{(2,s),(3,g-3)}(x'_{g-1,0}, x'_{3,1})$. By Lemma 5.1 (i), we obtain $\tilde{\partial}(y_{0,j'}, y_{5,j'}) = \tilde{\partial}(x_{-1,1}, x'_{3,1})$. Since $k_{2+2j',s'_j} = 1$ and $k_{\tilde{d}_2} = 6$, from Lemma 2.1 (vi), one has $s'_j > 5$. Observe $x_{0,1} \in P_{(1,g-1),\tilde{d}_2}(x_{-1,1}, x'_{3,1})$ and $x_{3,1} \in P_{(4,g-4),(1,g-1)}(x_{-1,1}, x'_{3,1})$. By the claim and Lemma 5.1 (i), we get $(x_{-1,1}, u_2) \in \Gamma_{\tilde{d}_2}$ and $(x_{0,0}, u_2) \in \Gamma_{\tilde{h}_2}$.

Since $\Gamma_{2,s}\Gamma_{1,g-1} = \{\Gamma_{3,s-1}\}$, from the claim, we get $u_1, x'_{2,0} \in P_{\tilde{f}_2,(1,g-1)}(x_{0,0}, u_2)$. By Lemma 5.1 (i), we obtain $p_{(1,g-1),\tilde{f}_2}^{\tilde{h}_2} = 2$ and $\tilde{\partial}(y_{1,j}, y_{4,j}) = \tilde{\partial}(y_{0,j}, y_{3,j}) = \tilde{f}_2$. It follows from Lemma 2.1 (vi) that $k_{\tilde{h}_2} = 3$. Pick a vertex $v_1 \in P_{(1,g-1),\tilde{f}_2}(y_{0,j}, y_{4,j}) \setminus \{y_{1,j}\}$ and a path $(v_1, v_2, v_3, y_{4,j})$. Observe $x_{1,1} \in P_{(1,g-1),\tilde{f}_2}(x_{0,1}, x'_{3,1})$ and $x_{3,1} \in P_{(3,g-3),(1,g-1)}(x_{0,1}, x'_{3,1})$. By Lemma 5.3, Lemma 2.1 (iii) and the commutativity of Γ , one has $y_{1,j}, v_1 \notin \Gamma_{\tilde{d}_2}(y_{5,j})$. The fact $x_{2,1} \in P_{(2,s),(2,g-2)}(x_{1,0}, x'_{3,1})$ and $(x_{1,1}, x'_{3,1}) \in \Gamma_{\tilde{f}_2}$ imply $v_1, y_{1,j} \in P_{(1,g-1),\tilde{h}_2}(y_{0,j}, y_{5,j})$. Since $(y_{0,j}, y_{3,j}) \in \Gamma_{\tilde{f}_2}$ and $p_{(1,g-1),(2,g-2)}^{\tilde{f}_2} = 1$, we obtain $v_3 \neq y_{3,j}$. By $p_{(2,s),(g-1,1)}^{(1,g-1)} = 1$, $p_{\tilde{\partial}(y_{0,j}, y_{5,j}), (g-1,1)}^{\tilde{h}_2} = 1$. From Lemma 2.1 (ii), we get $k_{\tilde{\partial}(y_{0,j}, y_{5,j})} = 1$ and $p_{(1,g-1),\tilde{h}_2}^{\tilde{\partial}(y_{0,j}, y_{5,j})} = 3$. Thus, $s_j = 5$.

Since $k_{4,s_1} = 1$, from Lemma 5.2 (ii) and Lemma 2.1 (iii), one gets $\tilde{\partial}(x_{0,0}, x_{3,2}) = (5, s_1 - 1)$. By $x_{2,2} \in P_{(4,s_1),(1,g-1)}(x_{0,0}, x_{3,2})$ and Lemma 2.1 (i), we have $j = 0$ and $g = 6$. Since $\Gamma_{2,5}\Gamma_{2,4} = \{\Gamma_{4,3}\}$, we obtain $k_{4,3} = 6$, contrary to $k_{3,4} = k_{3,s-1} = 3$.

Case 3. $t \geq 3$.

Observe $x_{1,1} \in \Gamma_{2,s}(x_{0,0}) \cap \Gamma_{g-3,3}(x_{4,1}) \cap \Gamma_{g-3,3}(x'_{3,1})$. By Lemma 2.1 (iii) and Lemma 5.2 (ii), we have $\partial(x_{0,0}, x'_{3,1}) = 5$, which implies $\partial(x_{0,0}, x_{4,2}) = 6$ and $\partial(x_{0,0}, x_{5,3}) < 8$ from Lemma 5.4.

Since $k_{4,s_1} = 1$ and $\partial(x_{0,0}, x_{3,2}) = 5$, from Lemma 2.1 (iii), we get $\partial(x_{0,0}, x_{3,3}) = 6$. By $x_{4,2}, x'_{3,2} \in \Gamma_{2,g-2}(x_{2,2})$, one has $\partial(x_{0,0}, x'_{3,2}) = 6$, which implies $\partial(x_{0,0}, x_{4,3}) =$

7. Observe that $x_{2,2} \in P_{(4,s_1),(3,g-3)}(x_{0,0}, x_{5,2})$ and $x_{2,2} \in P_{(4,s_1),(3,g-3)}(x_{0,0}, x'_{4,2})$. By Lemma 2.1 (iii) and $\partial(x_{0,0}, x_{5,3}) < 8$, we obtain $\partial(x_{0,0}, x_{5,2}) < 7$.

If $t \geq 5$, then $(x_{0,0}, x'_{4,0}) \in \Gamma_{5,g-5}$; if $t = 4$, then $(x_{0,0}, x'_{4,0}) \in \Gamma_{\tilde{f}_4}$; if $t = 3$, then $(x_{0,0}, x'_{4,0}) \in \Gamma_{\tilde{d}_3}$. By Lemma 5.1 (ii) or Lemma 5.4, we have $\partial(x_{0,0}, x'_{4,0}) = 5$. Since $\partial(x_{0,0}, x_{5,0}) = \partial(x_{0,0}, x_{4,1}) = 5$, we have $\partial(x_{0,0}, x_{5,1}) = 6$. Hence, $\partial(x_{0,0}, x'_{4,1}) < 6$.

Since $x_{2,1} \in P_{(2,s),(3,g-3)}(x_{1,0}, x'_{4,1})$, from Lemma 2.1 (iii), we get $\partial(x_{1,0}, x'_{4,1}) = \partial(x_{0,0}, x_{4,1}) = 5$. By $\tilde{\partial}(x_{0,0}, x'_{4,0}) = \tilde{\partial}(x_{0,1}, x'_{4,1})$, one has $\partial(x'_{0,0}, x'_{4,1}) < 5$, which implies $t = 3$ or 4, contrary to Lemma 5.4 or Lemma 5.1 (iii). \square

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Table 1: Two way distance of digraphs in Theorem 1.1

Γ	Conditions	$\tilde{\partial}((0,0), (a,b))$ with $(a,b) \neq (0,0)$
(vi)	$a = 0$	(g, g)
	$a \neq 0$	$(\hat{a}, g - \hat{a})$
(vii)	$b = 0$	$(\min\{\hat{a}, 2n - 2\hat{a}\}, \min\{n - \hat{a}, 2\hat{a}\})$
	$a = 0$	$(\min\{\hat{b}, 2n - 2\hat{b}\}, \min\{n - \hat{b}, 2\hat{b}\})$
	$a = b$	$(\min\{n - \hat{a}, 2\hat{a}\}, \min\{\hat{a}, 2n - 2\hat{a}\})$
	$\hat{a} > \hat{b} > 0$	(h_0, l_0)
	$\hat{b} > \hat{a} > 0$	(h_1, l_1)
(viii)	$-n < \hat{b} - \hat{a} < 0$	$(\min\{3n - 3\hat{a} + \hat{b}, 3\hat{a} - 2\hat{b}\}, \min\{3\hat{a} - \hat{b}, 3n + 2\hat{b} - 3\hat{a}\})$
	$0 \leq \hat{b} - \hat{a} < n$	$(\hat{b}, \max\{3\hat{a} - \hat{b}, 2\hat{b} - 3\hat{a}\})$
	$n \leq \hat{b} - \hat{a} < 2n$	$(\max\{3n + 3\hat{a} - 2\hat{b}, \hat{b} - 3\hat{a}\}, 3n - \hat{b})$
	$2n \leq \hat{b} - \hat{a} < 3n$	$(\min\{6n + 3\hat{a} - 2\hat{b}, \hat{b} - 3\hat{a}\}, \min\{3n + 3\hat{a} - \hat{b}, 2\hat{b} - 3\hat{a} - 3n\})$

For any element a in a residue class ring, we assume that \hat{a} denotes the minimum nonnegative integer in a . For $i = 0, 1$, let

$$\begin{aligned}
 h_i &= \min\{\hat{a} + \hat{b}, (i+1)n + \hat{a} - 2\hat{b}, (2-i)n - 2\hat{a} + \hat{b}\}, \\
 l_i &= \min\{2n - \hat{a} - \hat{b}, (1-i)n - \hat{a} + 2\hat{b}, in - \hat{b} + 2\hat{a}\}.
 \end{aligned}$$